

Regular patterns, substitutes, Feynman categories and operads

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ABSTRACT. We show that the regular patterns of Getzler (2009) form a 2-category biequivalent to the 2-category of substitutes of Day and Street (2003), and that the Feynman categories of Kaufmann and Ward (2013) form a 2-category biequivalent to the 2-category of coloured operads. These biequivalences induce equivalences between the corresponding categories of algebras. There are three main ingredients in establishing these biequivalences. The first is a strictification theorem (exploiting Power’s General Coherence Result) which allows to reduce to the case where the structure maps are identity-on-objects functors and strict monoidal. Second, we subsume the Getzler and Kaufmann–Ward hereditary axioms into the notion of Guitart exactness, a general condition ensuring compatibility between certain left Kan extensions and a given monad, in this case the free-symmetric-monoidal-category monad. Finally we set up an adjunction between substitutes and what we call pinned symmetric monoidal categories, from which the results follow as a consequence of the fact that the hereditary map is precisely the counit of this adjunction.

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0. Introduction and overview of results

A proliferation of operad-related structures have seen the light in the past decades, such as modular and cyclic operads, properads, and props. Work of many people has sought to develop categorical formalisms covering all these notions on a common footing, and in particular to describe adjunctions induced by the passage from one type of structure to another as a restriction/Kan extension pair [2, 3, 5, 7, 9, 11, 14, 23, 30]. For the line of development of the present work, the work of Costello [9] was especially inspirational: in order to construct the modular envelope of a cyclic operad, he presented these notions as symmetric monoidal functors out of certain symmetric

monoidal categories of trees and graphs, and arrived at the modular envelope as a left Kan extension corresponding to the inclusion of one symmetric monoidal category into the other. Unfortunately it is not clear from this construction that the resulting functor is even symmetric monoidal. The problem was addressed by Getzler [14] by identifying a condition needed for the construction to work: he introduced the notion of a “regular pattern” (cf. 0.1 below), which includes a condition formulated in terms of Day convolution, and which guarantees that constructions like Costello’s will work. However, his condition is not always easy to verify in practice. Meanwhile, Markl [23], and later Borisov and Manin [7], studied general notions of graph categories, designed with generalised notions of operad in mind, and isolated in particular a certain hereditary condition, which has also been studied by Mellies and Tabareau [24] within a different formalism (cf. 3.2 below). This condition found a comma-category formulation in the recent work of Kaufmann and Ward [18], being the essential axiom in their notion of “Feynman category”, cf. 0.2 below.

Kaufmann and Ward notice that the hereditary axiom is closely related to the Day-convolution Kan extension property of Getzler, and provide an easy-to-check condition under which the envelope construction (and other constructions given by left Kan extensions) work. Their work is the starting point for our investigations.

Another common generalisation of operads and symmetric monoidal categories is the notion of substitute proposed earlier by Day and Street [11]. Their interest came from the study of a nonstandard convolution construction introduced by Bakalov, D’Andrea, and Kac [1].

It turns out that Getzler’s notion of regular pattern is biequivalent to the notion of substitute of Day and Street [11], and that the Kaufmann–Ward notion of Feynman category is biequivalent to the notion of (coloured) operad. These two statements are the main results of this article. To explain the meaning of this equivalence observe that regular patterns and Feynman categories form 2-categories and, hence, the notion of biequivalence is appropriate here. It is less standard that also substitutes and operads naturally form 2-categories (see sections 5.1 and 5.6), and so we can speak about biequivalences as claimed. We also show that for both equivalences the notions of algebra agree, and study the hereditary condition from a categorical perspective, relating it to the exactness condition considered in [3, 30]. In a broader perspective, these results can be seen as part of a dictionary between symmetric-monoidal-category, operadic and 2-monadic approaches to operad-like structures, which goes back to the origins of operad theory, cf. Chapter 2 of Boardman–Vogt [6], and Kelly’s paper on clubs [19].

Part of the structure of Feynman category is an explicit groupoid, which one might think of as a “groupoid of colours”, in contrast to the set of colours of an operad. Somewhat surprisingly, perhaps, this groupoid turns out to be available already in the usual notion of operad, namely as the groupoid of invertible unary operations.

In the present paper, for the sake of focusing on the principal ideas, we work only over the category of sets. For the enriched setting, we refer to Caviglia [8], who independently has established an enriched version of the equivalence between Feynman categories and operads.

We proceed to state our main result, and sketch the ingredients that go into its proof.

For C a category, we denote by SC the free symmetric monoidal category on C .

0.1. Definition of regular pattern. (Getzler [14]) A *regular pattern* is a symmetric strong monoidal functor $\tau : SC \rightarrow M$ such that

- (1) τ is essentially surjective

- (2) the induced functor of presheaves $\tau^* : \widehat{M} \rightarrow \widehat{SC}$ is strong monoidal for the Day convolution tensor product.

0.2. Definition of Feynman category. (Kaufmann–Ward [18]) A *Feynman category* is a symmetric strong monoidal functor $\tau : SC \rightarrow M$ such that

- (1) C is a groupoid
- (2) τ induces an equivalence of groupoids $SC \simeq M_{\text{iso}}$
- (3) τ induces an equivalence of groupoids $S(M \downarrow C)_{\text{iso}} \simeq (M \downarrow M)_{\text{iso}}$.

0.3. The hereditary condition. Getzler’s definition is staged in the enriched setting. Kaufmann and Ward also give an enriched version called *weak Feynman category* ([18], Definition 4.2 and Remark 4.3) which over **Set** reads as follows:

$\tau : SC \rightarrow M$ is an essentially surjective symmetric strong monoidal functor, and the following important *hereditary* condition holds (formulated in more detail in 3.2): For any $x_1, \dots, x_m, y_1, \dots, y_n \in C$, the natural map given by tensoring

$$\sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\bigotimes_{i \in \alpha^{-1}(j)} \tau x_i, \tau y_j) \longrightarrow M(\bigotimes_{i \in \underline{m}} \tau x_i, \bigotimes_{j \in \underline{n}} \tau y_j)$$

is a bijection. (They recognise that this weak notion is ‘close’ to Getzler’s notion of regular pattern but do not prove that it is actually equivalent. In fact this condition does not really play a role in the developments in [18].)

The hereditary condition is natural from a combinatorial viewpoint where it says that every morphism splits into a tensor product of ‘connected’ morphisms. We shall see (5.11) that it is exactly the condition that the counit for the substitute Hermida adjunction is fully faithful.

0.4. Operads and substitutes. By *operad* we mean coloured symmetric operad in **Set**. We refer to the colours as objects. The notion of substitute was introduced by Day and Street [11], as a general framework for substitution in the enriched setting. Our substitutes are their symmetric substitutes, cf. also [4], whose appendix constitutes a concise reference for the basic theory of substitutes. A quick definition is this (cf. [12, 6.3]): a *substitute* is an operad equipped with an identity-on-objects operad morphism from a category (regarded as an operad with only unary operations).

We can now state the main theorem:

Theorem. (Cf. Theorem 5.13 and Theorem 5.15.) *There is a biequivalence between the bicategory of substitutes and the bicategory of regular patterns. It restricts to a biequivalence between the bicategory of operads and the bicategory of Feynman categories.*

The biequivalence means that when going back and forth, not an isomorphic object is obtained, but only an equivalent one. The equivalence is only a question of strictification: a key ingredient in the proof is to show that every regular pattern is equivalent to a strict one, and the main theorem is really a 1-equivalence between these strict regular patterns and substitutes. It should be observed that equivalent regular patterns have equivalent algebras (5.18).

We briefly run through the main ingredients of the proof, and outline the contents of the paper.

In Section 1, we show that regular patterns and Feynman categories can be strictified. Both notions concern a symmetric strong monoidal functor $\tau : SC \rightarrow M$ where SC is the free symmetric monoidal category on a category C , and in particular is strict. The main result is this:

Proposition. (Cf. Corollary 1.7.) *Every essentially surjective symmetric strong monoidal functor $SC \rightarrow M$, is equivalent to one $SC \rightarrow M'$, for which M' is a symmetric strict monoidal category, and $SC \rightarrow M'$ is strict monoidal and identity-on-objects.*

which is a consequence of Power's coherence result [25], recalled in Appendix A. Since the notions of regular pattern and Feynman category are invariant under monoidal equivalence, we may as well work solely with the strict case, which will facilitate the arguments greatly, and highlight the essential features of the notions, over the subtleties of having coherence isomorphisms everywhere.

The next step, which makes up Section 2, is to put Getzler's condition (2) into the context of Guitart exactness.

0.5. Guitart exactness. Guitart [15] introduced the notion of exact square: they are those squares that pasted on top of a pointwise left Kan extension again gives a pointwise left Kan extension. A morphism of T -algebras for a monad T is exact when the algebra morphism coherence square is exact. We shall need this notion only in the case where T is the free symmetric monoidal category monad on **Cat**: it thus concerns symmetric monoidal functors.

Proposition. (Cf. 2.11.) *The following are equivalent for a symmetric colax monoidal functor $\tau : S \rightarrow M$.*

- (1) τ is Guitart exact
- (2) left Kan extension of Yoneda along τ is strong monoidal
- (3) left Kan extension of any strong monoidal functor along τ is again strong monoidal
- (4) $\tau^* : \widehat{M} \rightarrow \widehat{S}$ is strong monoidal
- (5) a certain category of factorisations is connected (c.f. Lemmas 2.4 and 3.4)

In the special case of interest to us we thus have

Corollary. *For a symmetric strong monoidal functor $\tau : SC \rightarrow M$, axiom (2) of being a regular pattern is equivalent to being exact.*

In Section 3 we analyse the hereditary condition, which turns out to be equivalent to a special case of Guitart exactness. For the special case pertaining to Feynman categories, we thus obtain

Proposition. (Cf. 3.15.) *An essentially surjective symmetric strong monoidal functor $\tau : SC \rightarrow M$ is exact if and only if it satisfies the hereditary condition.*

Axiom (3) of the notion of Feynman category of Kaufmann and Ward [18], the equivalence of comma categories $S(M \downarrow C)_{\text{iso}} \xrightarrow{\sim} (M \downarrow M)_{\text{iso}}$, is of a slightly different flavour to the other related conditions (and in particular, does not seem to carry over to the enriched context). While it is implicit in [18] that this condition is essentially equivalent to the hereditary condition, the relationship is actually involved enough to warrant a detailed proof, which makes up our Section 4.

The outcome is the following result, essentially proved by Kaufmann and Ward [18].

Corollary. *A Feynman category is a special case of a regular pattern, namely such that C is a groupoid and $SC \rightarrow M_{\text{iso}}$ is an equivalence.*

After these reduction steps, the two notions read as follows:

A regular pattern is essentially a symmetric strict monoidal functor $\tau : SC \rightarrow M$ which is bijective on objects and satisfies the hereditary condition. A Feynman category is a regular pattern for which

furthermore C is a groupoid and the functor τ is fully faithful on isomorphisms (so that altogether $SC \rightarrow M_{\text{iso}}$ is an isomorphism of groupoids).

Having made these reductions, in Section 5 we finally obtain the results referred to in the title by setting up some variations of the symmetric Hermida adjunction [16] between symmetric monoidal categories and operads:

0.6. Pinned symmetric monoidal categories and pinned operads. A *pinned symmetric monoidal category* is a symmetric monoidal category M equipped with a symmetric strong monoidal functor $SC \rightarrow M$ (where SC is the free symmetric monoidal category on some category C). Hence regular patterns and Feynman categories are examples of pinned symmetric monoidal categories. Similarly, a *pinned operad* is defined to be an operad equipped with a functor from a category, viewed as an operad with only unary operations. Substitudes are thus pinned operads for which the structure map is identity-on-objects. The latter condition exhibits substitudes as a coreflective subcategory of pinned operads.

0.7. The substitute Hermida adjunction. Our main result will follow readily from the following variation on the Hermida adjunction, which goes between pinned symmetric monoidal categories and substitudes, via pinned operads:

$$(1) \quad pSMC \xrightleftharpoons{\perp} pOpd \xrightleftharpoons{\perp} Subst$$

The right adjoint takes a pinned symmetric monoidal category $SC \rightarrow M$ to the substitute

$$C \rightarrow \text{End}(M)|C$$

the endomorphism operad on M , base-changed to C . It is an important feature of substitudes (not enjoyed by operads) that they can be base-changed along functors.

The left adjoint in (1) takes a substitute $C \rightarrow P$ to the pinned symmetric monoidal category $SC \rightarrow FP$, where FP is the free symmetric monoidal category on P as in the ordinary Hermida adjunction: the objects of FP are finite sequences of objects in P , and its arrows from sequence x_1, \dots, x_m to sequence y_1, \dots, y_n are given by

$$FP(\mathbf{x}, \mathbf{y}) := \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} P((x_i)_{i \in \alpha^{-1}(j)}, y_j).$$

The left adjoint is now shown to be fully faithful (5.9). Our key result characterises its image by determining where the counit is invertible:

Proposition. (Cf. Proposition 5.12.) *The counit ε_τ is a bijective-on-objects functor if and only if $\tau : SC \rightarrow M$ is bijective on objects, and it is a fully faithful functor if and only if the hereditary condition holds.*

Corollary. *The image of the left adjoint is the category of (strictified) regular patterns.*

In particular, this establishes the first part of the Main Theorem, the equivalence between substitudes and regular patterns.

There is a fully faithful functor from operads to substitudes, which sends an operad P to its canonical groupoid pinning $P_1^{\text{iso}} \rightarrow P$. We characterise its image in the category of regular patterns:

Proposition. (Cf. 5.15.) *Under the above equivalence, the subcategory of operads corresponds to (and therefore is equivalent to) the category of Feynman categories.*

To derive the biequivalence in the main theorem, note that both regular patterns and strict regular patterns form bicategories, and that the inclusion of the strict into the general is part of a biequivalence. On the other hand, the equivalence of 1-categories obtained from the adjunction can easily be souped up to a 2-equivalence: both sides of the equivalence have natural notions of 2-cells, and these can be checked to match up as required.

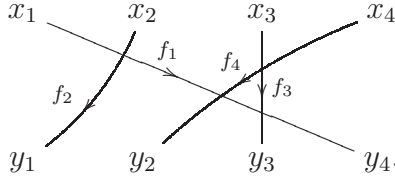
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1. Strictification of regular patterns

1.1. The free symmetric monoidal category. The free symmetric monoidal category \mathbf{SC} on a category C has the following explicit description. The objects of \mathbf{SC} are the finite sequences of objects of C . A morphism is of the form

$$(\rho, (f_i)_{i \in \underline{n}}) : (x_i)_{i \in \underline{n}} \longrightarrow (y_i)_{i \in \underline{n}}$$

where $\rho \in \Sigma_n$ is a permutation, and for $i \in \underline{n} = \{1, \dots, n\}$, $f_i : x_i \rightarrow y_{\rho i}$. Intuitively such a morphism is a permutation labelled by arrows of C , as in



For further details, see the Appendix, where it is explained and exploited that \mathbf{S} underlies a 2-monad on \mathbf{Cat} . It will be important that \mathbf{SC} is actually a symmetric *strict* monoidal category.

1.2. Gabriel factorisation. Given a functor $F : C \rightarrow D$, its factorisation into an identity-on-objects followed by a fully faithful functor is referred to as the *Gabriel factorisation* of F :

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ & \searrow \text{i.o.} & \nearrow \text{f.f.} \\ & D' & \end{array}$$

Proposition 1.3. *Let $F : S \rightarrow M$ be a symmetric strong monoidal functor, and assume that S is a symmetric strict monoidal category. Then for the Gabriel factorisation*

$$\begin{array}{ccc} S & \xrightarrow{F} & M \\ & \searrow G & \nearrow H \\ & M' & \end{array}$$

there is a canonical symmetric strict monoidal structure on M' for which G is a symmetric strict monoidal functor, and H is canonically symmetric strong monoidal.

The proof, relegated to the Appendix, exploits the general coherence result of Power [25]. A direct proof is somewhat subtle because the monoidal structure on M' is constructed as a mix of the monoidal structures on S and on M , and it is rather cumbersome to check that the trivial associator defined on M' is actually natural. Instead following Power's approach gives an elegant abstract proof, which exploits the following easily checked facts: (1) the Gabriel factorisation has a 2-dimensional aspect where isomorphisms can always be shifted right in the factorisation; and (2) S preserves this factorisation.

1.4. Pinned symmetric monoidal categories. The following terminology will be justified in Section 5, as part of further pinned notions. A *pinned symmetric monoidal category* is a symmetric monoidal category M equipped with a symmetric strong monoidal functor $\tau : SC \rightarrow M$ (where C is some category). A morphism $(C, \tau, M) \rightarrow (C', \tau', M')$ is a pair (F, G) where $F : C \rightarrow C'$ is a functor and $G : M \rightarrow M'$ is a symmetric strong monoidal functor, such that $\tau' \circ SF = G \circ \tau$. A 2-cell $(F, G) \rightarrow (F', G')$ is a pair (α, β) , where $\alpha : F \rightarrow F'$ is a natural transformation and $\beta : G \rightarrow G'$ is a monoidal natural transformation, such that $\tau' \circ S\alpha = \beta \circ \tau$.

1.5. Regular patterns. (Getzler [14]) A *regular pattern* is a symmetric strong monoidal functor $\tau : SC \rightarrow M$ such that

- (1) τ is essentially surjective
- (2) the induced functor of presheaves $\tau^* : \widehat{M} \rightarrow \widehat{SC}$ is strong monoidal for the Day convolution tensor product.

Regular patterns form a 2-category, namely the full sub-2-category of that of pinned symmetric monoidal categories spanned by the regular patterns.

1.6. Strict regular patterns. A pinned symmetric monoidal category (C, τ, M) is called *strict* when M is a symmetric *strict* monoidal category, and τ is a symmetric strict monoidal functor. A morphism $(F, G) : (C, \tau, M) \rightarrow (C', \tau', M')$ is *strict* when G is a symmetric strict monoidal functor. The locally full sub-2-category spanned by the strict objects and strict morphisms is denoted **pSMC**. Strict regular patterns and strict morphisms thereof are defined in the same way, and the corresponding 2-category is denoted **RPat**.

Corollary 1.7. *Every regular pattern $\tau : SC \rightarrow M$ is equivalent to a strict one, $\tau' : SC \rightarrow M'$, with the same category C . In particular, the inclusion functor of **RPat** into the 2-category of all regular patterns is a biequivalence.*

PROOF. This follows from Proposition 1.3 by noting that since τ is already essentially surjective by assumption, the fully faithful part of the factorisation will actually be an equivalence. \square

1.8. Algebras. Let W be a symmetric monoidal category. An *algebra* for a regular pattern $\tau : SC \rightarrow M$ in W , is a symmetric strong monoidal functor $M \rightarrow W$. With morphisms of algebras given by monoidal natural transformations, one has a category $\text{Alg}_\tau(W)$ of algebras of (C, τ, W) in W . A morphism of regular patterns $(C, \tau, M) \rightarrow (C', \tau', M')$ induces a functor $\text{Alg}_{\tau'}(W) \rightarrow \text{Alg}_\tau(W)$ by precomposition. The following proposition is now clear.

Proposition 1.9. *Equivalent regular patterns have equivalent categories of algebras.* \square

Together with Corollary 1.7, this justifies considering only strict regular patterns, as we do for the remainder of this article. This facilitates comparison with substitutes and operads.

2. Guitart exactness

In this section and the next we show how the main axioms in the definitions of regular pattern and Feynman category can be subsumed in the theory of Guitart exactness. An important aspect of Guitart exactness is to serve as a criterion for pointwise left Kan extensions to be compatible with algebraic structures. This direction of the theory is developed rather systematically in [30] in the abstract setting of a 2-monad on a 2-category with comma objects. The interesting case for the present purposes is the case of the free-symmetric-monoidal-category monad on **Cat**, and the issue is then under what circumstances left Kan extensions are symmetric monoidal functors.

We write $\hat{A} := [A^{\text{op}}, \mathbf{Set}]$ for the category of presheaves, and $y_A : A \rightarrow \hat{A}$ for the Yoneda embedding.

2.1. Exact squares. A 2-cell in **Cat** of the form

$$(2) \quad \begin{array}{ccc} P & \xrightarrow{q} & B \\ p \downarrow & \Rightarrow \phi & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

(called a *lax square*) is *exact in the sense of Guitart* [15] when for any natural transformation ψ which exhibits l as a pointwise left Kan extension of h along f , the composite

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ p \downarrow & \Rightarrow \phi & \downarrow g \\ A & \xrightarrow{f} & C \\ \searrow h & \Rightarrow \psi & \swarrow l \\ & V & \end{array}$$

exhibits lg as a pointwise left Kan extension of hp along q .

Suppose that in this situation A is locally small and f is admissible in the sense [26, 27] that $C(fa, c)$ is small for all $a \in A$ and $c \in C$. One thus has the functor $C(f, 1) : C \rightarrow \hat{A}$ given on objects by $c \mapsto C(f(-), c)$, and the effect on arrows of the functor f can be organised into a natural transformation

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \searrow y_A & \Rightarrow \chi^f & \swarrow C(f, 1) \\ & \hat{A} & \end{array}$$

which exhibits $C(f, 1)$ as a pointwise left Kan extension of y_A along f (see e.g. [27] Example 3.3).

Lemma 2.2. (Cf. Guitart [15].) *A lax square (2) in which A and P are small and f is admissible is exact if and only if the composite*

$$(3) \quad \begin{array}{ccc} P & \xrightarrow{q} & B \\ p \downarrow & \Rightarrow \phi & \downarrow g \\ A & \xrightarrow{f} & C \\ \searrow y_A & \Rightarrow \chi^f & \swarrow C(f, 1) \\ & \hat{A} & \end{array}$$

exhibits $C(f, 1) \circ g$ as a pointwise left Kan extension of $y_A \circ p$ along q . \square

When $f = y_A$, the 2-cell χ^f is the identity, and we get the following.

Corollary 2.3. *If P and A are small and*

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ p \downarrow & \Rightarrow \phi & \downarrow g \\ A & \xrightarrow{y_A} & \hat{A} \end{array}$$

exhibits g as a pointwise left Kan extension of $y_A \circ p$ along q , then ϕ is exact. \square

Exact squares can be recognised in elementary terms in the following way. First given $a \in A$, $b \in B$ and $\gamma : fa \rightarrow gb$ we denote by $\text{Fact}_\phi(a, \gamma, b)$ the following category. Its objects are triples (α, x, β) where $x \in P$, $\alpha : a \rightarrow px$ and $\beta : qx \rightarrow b$, such that $g(\beta)\phi_x f(\alpha) = \gamma$. Informally, such an object is a “factorisation of γ through ϕ ”. A morphism $(\alpha_1, x_1, \beta_1) \rightarrow (\alpha_2, x_2, \beta_2)$ of such is an arrow $\delta : x_1 \rightarrow x_2$ such that $p(\delta)\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 q(\delta)$. Identities and compositions are inherited from P .

Lemma 2.4. [15] *A lax square (2) in \mathbf{Cat} is exact iff for all $a \in A$, $b \in B$, and $\gamma : fa \rightarrow gb$, the category $\text{Fact}_\phi(a, \gamma, b)$ defined above is connected.*

2.5. Exact monoidal functors. In the usual nullary-binary way of writing tensor products in monoidal categories, a *symmetric colax monoidal functor* $f : A \rightarrow B$ has coherence morphisms of the form

$$\bar{f}_0 : fI \longrightarrow I \quad \bar{f}_{X,Y} : f(X \otimes Y) \longrightarrow fX \otimes fY$$

(in which “ I ” denotes the unit of either A or B) which are required to satisfy axioms that express compatibility with the coherences which define the symmetric monoidal structures on A and B . Equivalently one can regard a symmetric colax monoidal structure on f as comprising coherence morphisms

$$\bar{f}_{X_1, \dots, X_n} : f(X_1 \otimes \dots \otimes X_n) \longrightarrow f(X_1) \otimes \dots \otimes f(X_n)$$

for each sequence (X_1, \dots, X_n) of objects of A , whose naturality is expressed by the fact that they are the components of a natural transformation

$$\begin{array}{ccc} \mathbf{S}A & \xrightarrow{\mathbf{S}(f)} & \mathbf{S}B \\ \otimes \downarrow & \Rightarrow \bar{f} & \downarrow \otimes \\ A & \xrightarrow{f} & B. \end{array}$$

We say that f is *exact* when this square is an exact square in the sense discussed above. In terms of the 2-monad \mathbf{S} , (f, \bar{f}) is a colax morphism of pseudo algebras, and in [30], the theory of exact colax morphisms of algebras is developed at the general level of a 2-monad on a 2-category with comma objects.

The following lemma is key to the interest in exactness in the present context. The result is a special case of Theorem 2.4.4 of [30]. A similar result is obtained in the context of proarrow equipments in [24] and in a double categorical setting in [22].

Lemma 2.6. [30] *Let $f : A \rightarrow B$ be an exact symmetric colax monoidal functor. Then for any lax symmetric monoidal functor $g : A \rightarrow C$ (with C assumed algebraically cocomplete), the pointwise left Kan extension $\text{lan}_f g$*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \quad \Rightarrow \quad \swarrow \text{lan}_f g & \\ & C & \end{array}$$

is again naturally lax symmetric monoidal. Furthermore, if g is strong, then so is $\text{lan}_f g$. \square

The condition that C is algebraically cocomplete (with respect to f) means first of all that it has enough colimits for the left Kan extension in question to exist, and second, that these colimits are preserved by the tensor product in each variable. More formally, whenever ψ exhibits h as a pointwise left Kan extension of g along f as on the left, then the composite on the right

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \quad \Rightarrow \quad \swarrow h & \\ & C & \end{array} \qquad \begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ & \searrow Sg \quad \Rightarrow \quad \swarrow Sh & \\ & SC & \\ & \downarrow \otimes & \\ & C & \end{array}$$

exhibits $\otimes \circ Sh$ as a pointwise left Kan extension of $\otimes \circ Sg$ along Sf .

2.7. Day convolution tensor product. It is well known that the symmetric monoidal structure on A extends essentially uniquely to one on \hat{A} , for which the tensor product is cocontinuous in each variable. This tensor product $\ast : S\hat{A} \rightarrow \hat{A}$ is called *Day convolution* [10]. It is folklore that the Day convolution tensor product can also be characterised as a pointwise left Kan extension as in the following result, which is nothing more than a translation of the universal property of convolution as expressed by Im and Kelly [17], in these terms.

Proposition 2.8. [17] *For A a small symmetric monoidal category, the Day convolution tensor product \ast on \hat{A} can be characterised as the pointwise left Kan extension of $y_A \circ \otimes$ along Sy_A ,*

$$(4) \quad \begin{array}{ccc} SA & \xrightarrow{Sy_A} & S\hat{A} \\ \otimes \downarrow & \xRightarrow{\bar{y}_A} & \downarrow \ast \\ A & \xrightarrow{y_A} & \hat{A}. \end{array}$$

Furthermore, this square (invertible since Sy_A is fully faithful) constitutes the coherence data making y_A a symmetric strong monoidal functor. Finally, the following universal property holds (usually taken as the defining property of the Day convolution tensor product): For any cocomplete symmetric monoidal category X , composition with y_A gives equivalences of categories

$$\text{Cocts} \mathbf{SMC}_c(\hat{A}, X) \simeq \mathbf{SMC}_c(A, X) \qquad \text{Cocts} \mathbf{SMC}(\hat{A}, X) \simeq \mathbf{SMC}(A, X).$$

Here \mathbf{SMC} is the 2-category of symmetric monoidal categories with symmetric strong monoidal functors, while \mathbf{SMC}_c has also symmetric colax monoidal functors. The prefixes Cocts indicate the full subcategories spanned by cocomplete symmetric monoidal categories whose tensor product preserves colimits in both variables.

PROOF. For any category C , we denote by \mathbf{MC} the free (strict) monoidal category on C . Explicitly \mathbf{MC} is the subcategory of \mathbf{SC} containing all the objects, but just the morphisms whose underlying permutation is an identity. The inclusions $i_C : \mathbf{MC} \rightarrow \mathbf{SC}$ are the components of a 2-natural transformation $i : \mathbf{M} \rightarrow \mathbf{S}$ which by the results of [28], conforms to the hypotheses of Proposition 4.6.2 of [30]. Thus for any functor $f : C \rightarrow D$, the corresponding naturality square of i on the left

$$\begin{array}{ccc} \mathbf{MC} & \xrightarrow{Mf} & \mathbf{MD} \\ i_C \downarrow & & \downarrow i_D \\ \mathbf{SC} & \xrightarrow{Sf} & \mathbf{SD} \end{array} \qquad \begin{array}{ccc} \mathbf{MA} & \xrightarrow{My_A} & \mathbf{M}\hat{A} \\ i_A \downarrow & & \downarrow i_{\hat{A}} \\ \mathbf{SA} & \xrightarrow{Sy_A} & \mathbf{S}\hat{A} \\ \otimes \downarrow & \xRightarrow{\bar{y}_A} & \downarrow * \\ A & \xrightarrow{y_A} & \hat{A} \end{array}$$

is exact, and so the composite square on the right exhibits $* \circ i_{\hat{A}}$ as a pointwise left Kan extension, and this functor has the same object map as $*$. Computing the left Kan extension on the left in the previous display as a coend in the usual way, one recovers the usual formula for the Day tensor product. Thus the result follows from [17]. \square

With Corollary 2.3, we arrive at the following.

Corollary 2.9. $y_A : (A, \otimes) \rightarrow (\hat{A}, *)$ is exact. \square

2.10. **Generalities.** For any functor $f : A \rightarrow B$ between small categories, we have the 2-cells

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{y_B} & \hat{B} \\ & \searrow y_A & \downarrow B(f,1) & \swarrow f^* & \\ & & \hat{A} & & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{y_A} & \hat{A} \\ f \downarrow & \lrcorner l^f & \downarrow f_! \\ B & \xrightarrow{y_B} & \hat{B} \end{array}$$

exhibiting $B(f, 1)$ as the pointwise left Kan extension of y_A along f , and f^* (restriction along f) as the pointwise left Kan extension of $B(f, 1)$ along y_B . Finally, l^f exhibits $f_!$ (the left adjoint to f^*) as the pointwise left Kan extension of $y_B \circ f$ along y_A . Note that l^f is an exact square by Corollary 2.3, and that both ι^f and l^f are invertible, since y_B and y_A are fully faithful.

Returning to our situation of a symmetric colax monoidal functor between small symmetric monoidal categories $(f, \bar{f}) : A \rightarrow B$, by Lemma 2.6 $f_!$ gets a symmetric colax monoidal structure from that of $y_B \circ f$, since y_A is exact by Proposition 2.8. The colax coherence datum $\bar{f}_!$ (which we don't explicitate here, and which is invertible if and only if \bar{f} is) induces, by taking mates via $f_! \dashv f^*$, the coherence 2-cell \bar{f}^* making f^* a lax monoidal functor. Moreover $B(f, 1)$ gets a unique monoidal structure making ι^f an invertible monoidal natural transformation. In the context just described we have the following alternative characterisations of exactness of the symmetric colax monoidal functor (f, \bar{f}) .

Proposition 2.11. *The following statements are equivalent for a colax symmetric monoidal functor $f : A \rightarrow B$ (assuming A small and f admissible).*

- (1) f is exact.
- (2) For any algebraically cocomplete symmetric monoidal category X and any symmetric strong monoidal functor $g : A \rightarrow X$, the pointwise left Kan extension of g along f is symmetric strong monoidal.

- (3) $B(f, 1) : B \rightarrow \hat{A}$ is symmetric strong monoidal.
 (4) $f^* : \hat{B} \rightarrow \hat{A}$ is symmetric strong monoidal.

PROOF. (1) \implies (2): By Lemma 2.6.

(2) \implies (3): By Proposition 2.8, y_A is strong monoidal. Since χ^f exhibits $B(f, 1)$ as a pointwise left Kan extension of y_A along f , we conclude by the assumption (2) that $B(f, 1)$ is strong monoidal.

(3) \implies (4): By Corollary 2.9, y_B is exact, and $B(f, 1)$ is strong monoidal by assumption. But ι^f exhibits f^* as the pointwise left Kan extension of $B(f, 1)$ along y_B , so again by Lemma 2.6 we conclude that f^* is strong monoidal.

(4) \implies (1): From [26, 27] the unit u of the adjunction $f_! \dashv f^*$ is the unique 2-cell satisfying the equation on the left

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xrightarrow{y_A} & \hat{A} \\ y_A \downarrow & \searrow \chi^{y_B f} & \downarrow f_! \\ \hat{A} & \xleftarrow{f^*} & \hat{B} \end{array} & = & \begin{array}{ccc} A & \xrightarrow{y_A} & \hat{A} \\ y_A \downarrow & \searrow \text{id} & \downarrow f_! \\ \hat{A} & \xleftarrow{f^*} & \hat{B} \end{array} \\
 & & \text{with } 1_{\hat{A}} \Rightarrow u
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 \begin{array}{ccc} S\hat{A} & \xrightarrow{Sf_!} & S\hat{B} \\ * \downarrow & \searrow \bar{f}_! & \downarrow * \\ \hat{A} & \xrightarrow{f_!} & \hat{B} \\ \uparrow u & & \downarrow f^* \\ \hat{A} & & \hat{A} \end{array} & = & \begin{array}{ccc} S\hat{A} & \xrightarrow{Sf_!} & S\hat{B} \\ 1_{S\hat{A}} \downarrow & \searrow Su & \downarrow * \\ S\hat{A} & \xleftarrow{Sf^*} & S\hat{B} \\ * \downarrow & \searrow \bar{f}^* & \downarrow * \\ \hat{A} & \xleftarrow{f^*} & \hat{B} \end{array}
 \end{array}$$

and $\bar{f}_!$ and \bar{f}^* determine each other uniquely by the equation on the right. Being the unit of an adjunction, Su is an absolute pointwise left Kan extension, and since \bar{f}^* is assumed to be invertible (4), the common composite of the equation on the right in the previous display exhibits $f^* \circ *_B$ as the pointwise left Kan extension of $*_A$ along $Sf_!$. Now paste on the left with \bar{y}_A which is a pointwise left Kan extension by Proposition 2.8. The resulting pointwise left Kan extension can be rewritten as follows:

$$\begin{array}{ccc}
 \begin{array}{ccccc} & & S\hat{A} & & \\ & \nearrow Sy_A & \downarrow * & \searrow Sf_! & \\ SA & \nearrow \bar{y}_A & \hat{A} & \nearrow \bar{f}_! & S\hat{B} \\ \otimes \downarrow & \nearrow y_A & \downarrow \text{id} & \searrow f_! & \downarrow * \\ A & \xrightarrow{f} & B & \xrightarrow{y_B} & \hat{B} \\ & \searrow y_A & \downarrow \chi^{y_B f} & \searrow f^* & \\ & & \hat{A} & & \end{array} & = & \begin{array}{ccccc} & & S\hat{A} & & \\ & \nearrow Sy_A & \downarrow * & \searrow Sf_! & \\ SA & \nearrow \bar{y}_A & \hat{A} & \nearrow \bar{f}_! & S\hat{B} \\ \otimes \downarrow & \nearrow y_A & \downarrow \text{id} & \searrow f_! & \downarrow * \\ A & \xrightarrow{f} & B & \xrightarrow{y_B} & \hat{B} \\ & \searrow y_A & \downarrow \chi^{y_B f} & \searrow f^* & \\ & & \hat{A} & & \end{array} & = & \begin{array}{ccccc} & & S\hat{A} & & \\ & \nearrow Sy_A & \downarrow * & \searrow Sf_! & \\ SA & \xrightarrow{Sf} & SB & \xrightarrow{Sy_B} & S\hat{B} \\ \otimes \downarrow & \nearrow \bar{f} & \downarrow \bar{y}_B & \searrow * & \\ A & \xrightarrow{f} & B & \xrightarrow{y_B} & \hat{B} \\ & \searrow y_A & \downarrow B(f,1) & \searrow \iota^f & \\ & & \hat{A} & & \end{array}
 \end{array}$$

and since $\mathbf{S}l^f$ is invertible, we conclude that already

$$\begin{array}{ccccc}
 \mathbf{S}A & \xrightarrow{\mathbf{S}f} & \mathbf{S}B & \xrightarrow{\mathbf{S}y_B} & \mathbf{S}\hat{B} \\
 \downarrow \otimes & \Downarrow \bar{f} & \downarrow \otimes & \Downarrow \bar{y}_B & \downarrow * \\
 A & \xrightarrow{f} & B & \xrightarrow{y_B} & \hat{B} \\
 \searrow \chi^f & & \downarrow B(f,1) & \swarrow \iota^f & \\
 & & \hat{A} & &
 \end{array}$$

$\begin{array}{ccc} y_A & & f^* \\ & \searrow & \swarrow \\ & \hat{A} & \end{array}$

exhibits $f^* \circ *_B$ as a pointwise left Kan extension of $y_A \circ \otimes_A$ along $\mathbf{S}y_B \circ \mathbf{S}f$. But already ι^f is a pointwise left Kan extension, and \bar{y}_B is an exact square, so the whole right-hand part of the diagram is a pointwise left Kan extension. Since furthermore $\mathbf{S}y_B$ is fully faithful, we can cancel that right-hand part away (e.g. by [20], Theorem 4.47), so in conclusion also

$$\begin{array}{ccc}
 \mathbf{S}A & \xrightarrow{\mathbf{S}f} & \mathbf{S}B \\
 \downarrow \otimes & \Downarrow \bar{f} & \downarrow \otimes \\
 A & \xrightarrow{f} & B \\
 \searrow \chi^f & & \swarrow B(f,1) \\
 & & \hat{A}
 \end{array}
 ,$$

is a pointwise left Kan extension. It now follows from Lemma 2.2 that f is exact. \square

Corollary 2.12. *For a symmetric monoidal functor $\tau : \mathbf{S}C \rightarrow M$, axiom (2) of being a regular pattern is equivalent to being exact.* \square

2.13. Morphisms of regular patterns. Recall that a morphism of regular patterns is a commutative square of strong symmetric monoidal functors

$$\begin{array}{ccc}
 \mathbf{S}C & \xrightarrow{\tau} & M \\
 \downarrow & & \downarrow g \\
 \mathbf{S}C' & \xrightarrow{\tau'} & M'.
 \end{array}$$

Proposition 2.14. *Every such $g : M \rightarrow M'$ is exact.*

PROOF. The free functor $\mathbf{S}C \rightarrow \mathbf{S}C'$ is exact by Corollary 4.6.6 of [30]. The two functors τ and τ' are exact by assumption, and τ is furthermore bijective on objects. It now follows from Lemma 2.15 that g is exact. \square

Lemma 2.15. *Given a commutative triangle of symmetric strong monoidal functors*

$$\begin{array}{ccc}
 S & \xrightarrow{f} & T \\
 \searrow u & & \swarrow g \\
 & S' &
 \end{array}$$

if f is exact, and u is exact and bijective on objects, then g is exact.

PROOF. By Proposition 2.11 it is enough to check that g^* is strong monoidal. Consider the corresponding triangle of pullback functors:

$$\begin{array}{ccc} \hat{S} & \xleftarrow{f^*} & \hat{T} \\ & \swarrow u^* \quad \searrow g^* & \\ & \hat{S}' & \end{array}$$

All three functors are lax monoidal; f^* and u^* are strong monoidal because of exactness. Furthermore u^* is monadic since u is bijective on objects, and so u^* is conservative. The monoidal coherences of f^* are invertible; but these are obtained by applying u^* to the lax coherence of g^* . Since u^* is conservative we can therefore conclude that already the coherences for g^* must be invertible. \square

3. The hereditary condition and exactness

In this section we analyse the hereditary condition of Kaufmann and Ward [18] and relate it to Guitart exactness. In Section 5 we shall see that the hereditary condition is one of two conditions characterising substitutes among pinned monoidal categories (Proposition 5.12).

3.1. Permutation-monotone factorisation. As in Section 1 for $n \in \mathbb{N}$, we denote by \underline{n} the linearly-ordered set $\{1, \dots, n\}$. Any function $\alpha : \underline{m} \rightarrow \underline{n}$ factors uniquely as

$$\alpha = \lambda_\alpha \circ \sigma_\alpha,$$

where $\sigma_\alpha : \underline{m} \xrightarrow{\sim} \underline{m}$ is a permutation that is monotone on the fibre $\alpha^{-1}(j)$ for each $j \in \underline{n}$, and $\lambda_\alpha : \underline{m} \rightarrow \underline{n}$ is monotone¹. With reference to $\alpha : \underline{m} \rightarrow \underline{n}$, if $(x_1, \dots, x_m) = (x_i)_{i \in \underline{m}}$ is a sequence of objects, we denote by $(x_i)_{\alpha i = j}$ the subsequence consisting of those entries whose index maps to j . The order is the induced order on the subset $\alpha^{-1}(j) \subset \underline{m}$.

3.2. The hereditary condition. A symmetric colax monoidal functor $\tau : SC \rightarrow M$ satisfies the *hereditary condition* when for all pairs of sequences $(x_i)_{i \in \underline{m}}$ and $(y_j)_{j \in \underline{n}}$ of objects of C , the function

$$h_{\tau, x, y} : \sum_{\alpha : \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i = j}, \tau y_j) \longrightarrow M(\tau(x_i)_{i \in \underline{m}}, \bigotimes_{j \in \underline{n}} \tau y_j)$$

which sends $(\alpha, (g_j)_{j \in \underline{n}})$ to the composite

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\tau \sigma_\alpha} \tau(x_{\sigma_\alpha^{-1} i})_{i \in \underline{m}} \xrightarrow{\bar{\tau}} \bigotimes_{j \in \underline{n}} \tau(x_i)_{\alpha i = j} \xrightarrow{\bigotimes_j g_j} \bigotimes_{j \in \underline{n}} \tau y_j$$

in M , is a bijection. (Note that $\bigotimes_{j \in \underline{n}} \tau(x_{\sigma_\alpha^{-1} i})_{\lambda_\alpha i = j} = \bigotimes_{j \in \underline{n}} \tau(x_i)_{\alpha i = j}$.) Note that the summation is taken over *arbitrary* functions $\alpha : \underline{m} \rightarrow \underline{n}$, not just monotone ones.

In the case where τ is strict (i.e. when $\bar{\tau}$ is the identity) one has $\tau(x_i)_{i \in \underline{m}} = \bigotimes_{i \in \underline{m}} \tau x_i$, and it may be more convenient to write $h_{\tau, x, y}$ as the function

$$\sum_{\alpha : \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\bigotimes_{\alpha i = j} \tau x_i, \tau y_j) \longrightarrow M(\bigotimes_{i \in \underline{m}} \tau x_i, \bigotimes_{j \in \underline{n}} \tau y_j)$$

which sends $(\alpha, (g_j)_{j \in \underline{n}})$ to the composite

$$\bigotimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sigma} \bigotimes_{i \in \underline{m}} \tau x_{\sigma^{-1} i} = \bigotimes_{j \in \underline{n}} \bigotimes_{\alpha i = j} \tau x_i \xrightarrow{\bigotimes_j g_j} \bigotimes_{j \in \underline{n}} \tau y_j.$$

¹This factorisation is not part of a factorisation system, but it is nevertheless very useful.

In less formal terms, the hereditary condition says that every morphism f of M as on the right

$$g_j : \bigotimes_{\alpha i=j} \tau x_i \longrightarrow \tau y_j \quad f : \bigotimes_{i \in \underline{m}} \tau x_i \longrightarrow \bigotimes_{j \in \underline{n}} \tau y_j$$

can be uniquely decomposed as a tensor product of morphisms g_j as on the left, modulo some symmetry coherence isomorphisms in M . A useful slogan for this is: ‘many-to-many maps decompose uniquely as a tensor product of many-to-one maps’; which expresses the operadic nature of this condition. The hereditary condition has been discovered independently by various people in different guises. While Kaufmann and Ward got it from Markl [23] via Borisov and Manin [7], it is also equivalent to the (operad case of the) ‘operadicity’ condition of Melliès and Tabareau [24, §3.2].

The first subgoal (Lemma 3.5) is to see that if an identity-on-objects symmetric strict monoidal functor satisfies the hereditary condition then it is exact. To this end we invoke a classical criterion for exactness in terms of a category of factorisations, going back to Guitart himself [15] in some form, and analysed in more detail in [30].

3.3. Factorisation category of a morphism $f : Fv \rightarrow \bigotimes_k w_k$. Consider a symmetric colax monoidal functor $F : V \rightarrow W$ (with coherences $\overline{F} : (\bigotimes_i v_i) \rightarrow \bigotimes_i Fv_i$). For each morphism in W of the form $f : Fv \rightarrow \bigotimes_k w_k$, we define the category of factorisations $\text{Fact}(f)$ as having objects diagrams of the form

$$\begin{array}{ccc} Fv & \xrightarrow{f} & \bigotimes_k w_k \\ & \searrow Fh & \nearrow \bigotimes_k g_k \\ & F(\bigotimes_k u_k) \longrightarrow \bigotimes_k F(u_k) & \end{array}$$

Morphisms in $\text{Fact}(f)$ are connecting maps (in V) between the u_k and the u'_k making a left-hand triangle commute already in V , and making the right hand triangle commute in W . (The middle square commutes by naturality of the coherence data.)

An immediate consequence of Lemma 2.4 is the following explicit characterisation of exact symmetric colax monoidal functors.

Lemma 3.4. *A symmetric colax monoidal functor $F : V \rightarrow W$ as above is exact if and only if for every $f : v \rightarrow \bigotimes_k w_k$ the category $\text{Fact}(f)$ is connected.*

When V is SC , and F is strict monoidal $\tau : SC \rightarrow M$, the description of $\text{Fact}(f)$ simplifies considerably. In this case $v = (x_i)_{i \in \underline{m}}$, and each u_k is a sequence $(z_i)_{i \in \underline{m}_k}$. Furthermore, since h is a map in SC it is a labelled permutation, so in particular the concatenated sequence $((z_i)_{i \in \underline{m}_k})_{k \in \underline{r}}$ is of the same length as \underline{m} , so altogether we can write a factorisation of $f : \tau(x_i)_{i \in \underline{m}} \rightarrow \bigotimes_{k \in \underline{r}} w_k$ as

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\tau(h)} \tau((z_i)_{\beta i=k})_{k \in \underline{r}} = \bigotimes_{k \in \underline{r}} \bigotimes_{\beta i=k} \tau z_i \xrightarrow{\bigotimes_{k \in \underline{r}} g_k} \bigotimes_{k \in \underline{r}} w_k.$$

Here $\beta : \underline{m} \rightarrow \underline{r}$ is monotone. Since h is a labelled permutation, we can keep the pure permutation part as a first factor, and then absorb the second factor (the ‘labels’) into the g_k on the right. Renaming those g_k accordingly, we arrive at a factorisation

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\sim} \tau((x_i)_{\beta i=k})_{k \in \underline{r}} = \bigotimes_{k \in \underline{r}} \bigotimes_{\beta i=k} \tau x_i \xrightarrow{\bigotimes_{k \in \underline{r}} g_k} \bigotimes_{k \in \underline{r}} w_k$$

Factorisations of this form we call *normalised*. Clearly every factorisation receives a morphism from its normalisation.

Lemma 3.5. *If a symmetric strong monoidal functor $\tau : SC \rightarrow M$ is essentially surjective and satisfies the hereditary condition, then it is exact.*

PROOF. By strictification, we can assume that τ is strict monoidal and identity on objects. We fix $f : \tau(x_i)_{i \in \underline{m}} = \otimes_{i \in \underline{m}} \tau x_i \rightarrow \otimes_{k \in \underline{r}} w_k$ and aim to show that $\text{Fact}(f)$ is connected. The identity-on-object condition means that each w_k is a tensor product of certain τy_j , say $w_k = \otimes_{\lambda i=k} \tau y_j$, where $\lambda : \underline{n} \rightarrow \underline{r}$ is monotone. Altogether,

$$\otimes_{k \in \underline{r}} w_k = \otimes_{k \in \underline{r}} \otimes_{\lambda i=k} \tau y_j = \otimes_{j \in \underline{n}} \tau y_j.$$

The map $f : \otimes_{i \in \underline{m}} \tau x_i \rightarrow \otimes_{k \in \underline{r}} w_k$ is now of the form

$$f : \otimes_{i \in \underline{m}} \tau x_i \rightarrow \otimes_{j \in \underline{n}} \tau y_j,$$

and by the hereditary condition we get a factorisation as

$$\otimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sim} \otimes_{j \in \underline{n}} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{j \in \underline{n}} s_j} \otimes_{j \in \underline{n}} \tau y_j,$$

which can also be written as

$$(5) \quad \otimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sim} \otimes_{k \in \underline{r}} \otimes_{\lambda j=k} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{k \in \underline{r}} (\otimes_{\lambda j=k} s_j)} \otimes_{k \in \underline{r}} (\otimes_{\lambda j=k} \tau y_j).$$

This we refer to as the standard factorisation of f . It is seen to be an object in $\text{Fact}(f)$ by putting $g_k = \otimes_{\lambda j=k} s_j$. In particular we have now shown that $\text{Fact}(f)$ is not empty.

Given now any other normalised object in $\text{Fact}(f)$:

$$(6) \quad \otimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sim} \otimes_{k \in \underline{r}} \otimes_{\beta i=k} \tau x_i \xrightarrow{\otimes_{k \in \underline{r}} g'_k} \otimes_{k \in \underline{r}} w_k,$$

where $\beta : \underline{m} \rightarrow \underline{r}$, we would like to connect it to the standard factorisation just constructed. To this end we apply the hereditary condition to each of the maps

$$g'_k : \otimes_{\beta i=k} \tau x_i \rightarrow w_k = \otimes_{\lambda j=k} \tau y_j.$$

This gives us a map $\alpha_k : \beta^{-1}(k) \rightarrow \lambda^{-1}(k)$, and all these maps assemble into a map $\alpha : \underline{m} \rightarrow \underline{n}$ such that $\lambda \circ \alpha = \beta$. With reference to these α_k , the hereditary condition gives us a normalised factorisation of g_k as

$$\otimes_{\beta i=k} \tau x_i \xrightarrow{\sim} \otimes_{\lambda j=k} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{\lambda j=k} s'_j} \otimes_{\lambda j=k} \tau y_j$$

so that altogether f factors as

$$\otimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sim} \otimes_{k \in \underline{r}} \otimes_{\beta i=k} \tau x_i \xrightarrow{\sim} \otimes_{k \in \underline{r}} \otimes_{\lambda j=k} \otimes_{\alpha i=j} \tau x_i \xrightarrow{\otimes_{k \in \underline{r}} \otimes_{\lambda j=k} s'_k} \otimes_{k \in \underline{r}} \otimes_{\lambda j=k} \tau y_j.$$

If we take the middle permutation to belong to the left-hand factor, then we obtain a factorisation of the standard shape (5), and by the uniqueness property in the hereditary condition, this must actually be *equal* to the standard factorisation (5), that is $s'_j = s_j$ for all $j \in \underline{n}$. On the other hand, if we let the middle permutation belong to the right-hand factor, we get precisely the given normalised factorisation (6). Hence the given normalised factorisation is connected to the standard factorisation. Since we already remarked that any factorisation is connected to its normalisation, we have altogether shown that $\text{Fact}(f)$ is connected. \square

An alternative proof follows from the results in Section 5: by Proposition 5.12, we may assume that τ is of the form $F(q)$, where $q : C \rightarrow P$ is a substitute. It can be checked directly that for any morphism of operads q , the symmetric monoidal functor $F(q)$ is exact. See [30] (Corollary 3.4.1) for a proof.

The second subgoal is to prove that exactness of a symmetric (colax) monoidal functor $\tau : \mathbf{SC} \rightarrow M$ implies the hereditary condition. The main idea is to characterise the hereditary condition directly in terms of exact squares, which we do in Corollary 3.12 below.

3.6. Notation. Until the end of this section, the symbols μ and η refer to the (C -component of the) multiplication and unit of the free-symmetric-monoidal-category monad \mathbf{S} .

An object in $\mathbf{S}^2 C$ is a sequence of sequences of objects in C . It is convenient to encode this as a pair $(\lambda, (x_i)_{i \in \underline{m}})$: here $(x_i)_{i \in \underline{m}}$ is the concatenated sequence, and $\lambda : \underline{m} \rightarrow \underline{n}$ is a *monotone* map indicating how $(x_i)_{i \in \underline{m}}$ arises from the n individual sequences, $(x_i)_{\lambda i = j}$. (In the notation $(\lambda, (x_i)_{i \in \underline{m}})$, the symbol \underline{m} indexing the sequence implicitly depends on λ , namely as its domain.)

Recall that a symmetric colax monoidal functor $\tau : \mathbf{SC} \rightarrow M$ has coherence datum

$$\begin{array}{ccc} \mathbf{S}^2 C & \xrightarrow{\mathbf{S}(\tau)} & \mathbf{S} M \\ \mu \downarrow & \xRightarrow{\bar{\tau}} & \downarrow \otimes \\ \mathbf{S} C & \xrightarrow{\tau} & M \end{array}$$

the components of which take the form

$$\bar{\tau}_{(\lambda, (x_i)_{i \in \underline{m}})} : \tau(x_i)_{i \in \underline{m}} \longrightarrow \bigotimes_{j \in \underline{n}} \tau(x_i)_{\lambda i = j}.$$

We proceed to reformulate the hereditariness of (a colax) $\tau : \mathbf{SC} \rightarrow M$ in terms of exact squares.

3.7. Naturality. The map in the hereditary condition,

$$h_{\tau, x, y} : \sum_{\alpha : \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i = j}, \tau y_j) \longrightarrow M(\tau(x_i)_{i \in \underline{m}}, \bigotimes_{j \in \underline{n}} \tau y_j),$$

is natural in the variables $x = (x_i)_{i \in \underline{m}}$ and $y = (y_j)_{j \in \underline{n}}$ in \mathbf{SC} in the following sense. Corresponding to the domain (the left-hand side) consider the functor $\mathcal{X}_\tau : \mathbf{SC} \rightarrow \widehat{\mathbf{SC}}$ defined on objects by

$$(\mathcal{X}_\tau(y_j)_{j \in \underline{n}})(x_i)_{i \in \underline{m}} = \sum_{\alpha : \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i = j}, \tau y_j),$$

and corresponding to the codomain (the right-hand side) consider the functor $\mathcal{Y}_\tau : \mathbf{SC} \rightarrow \widehat{\mathbf{SC}}$ defined on objects by

$$(\mathcal{Y}_\tau(y_j)_{j \in \underline{n}})(x_i)_{i \in \underline{m}} = M(\tau(x_i)_{i \in \underline{m}}, \bigotimes_{j \in \underline{n}} \tau y_j).$$

Now the functions $h_{\tau, x, y}$ of Definition 3.2 are the components of a natural transformation

$$h_\tau : \mathcal{X}_\tau \rightarrow \mathcal{Y}_\tau.$$

It is straightforward to check that \mathcal{Y}_τ is the composite functor

$$\mathbf{SC} \xrightarrow{\mathbf{S}\eta} \mathbf{S}^2 C \xrightarrow{\mathbf{S}\tau} \mathbf{S} M \xrightarrow{\otimes} M \xrightarrow{M(\tau, 1)} \widehat{\mathbf{SC}}.$$

3.8. The natural transformation γ . We denote by γ the comma square defining $\tau \downarrow \tau \eta$:

$$\begin{array}{ccc} \tau \downarrow \tau \eta & \xrightarrow{r} & C \\ q \downarrow & \xRightarrow{\gamma} & \downarrow \tau \eta \\ \mathbf{SC} & \xrightarrow{\tau} & M. \end{array}$$

Here $\eta : C \rightarrow \mathbf{S}C$ is the unit for \mathbf{S} ; it is a significant ingredient in the comma category, but we suppress it on objects. An object of $\tau \downarrow \tau \eta$ is a thus triple $((x_i)_{i \in \underline{n}}, g, y)$ where $g : \tau(x_i)_i \rightarrow \tau y$ is a morphism of M .

We shall need an explicit description also of $\mathbf{S}(\tau \downarrow \tau \eta)$: an object consists of a sequence of objects of $\tau \downarrow \tau \eta$; it is thus of the form $((x_{ij})_{i \in \underline{m}_j}, g_j, y_j)_{j \in \underline{n}}$, with each $g_j : \tau(x_{ij})_{i \in \underline{m}_j} \rightarrow y_j$ a morphism in M . Such data in turn amounts to $(\lambda, (x_i)_{i \in \underline{m}}, (g_j)_{j \in \underline{n}}, (y_j)_{j \in \underline{n}})$, where $\lambda : \underline{m} \rightarrow \underline{n}$ is monotone, and $g_j : \tau(x_i)_{\lambda i=j} \rightarrow y_j$ is in M . In these terms, morphisms of $\mathbf{S}(\tau \downarrow \tau \eta)$ are of the form

$$(7) \quad (\sigma, \rho, (a_i)_i, (b_j)_j) : (\lambda, (x_i)_{i \in \underline{m}}, (g_j)_{j \in \underline{n}}, (y_j)_{j \in \underline{n}}) \longrightarrow (\lambda', (x'_i)_{i \in \underline{m}}, (g'_j)_{j \in \underline{n}}, (y'_j)_{j \in \underline{n}})$$

where the permutations $\sigma \in \Sigma_m$, $\rho \in \Sigma_n$ satisfy $\lambda' \sigma = \rho \lambda$, and where $a_i : x_i \rightarrow x'_{\sigma i}$ and $b_j : y_j \rightarrow y_{\rho j}$ are maps in C such that $\tau(b_j)g_j = g'_j \tau(\sigma|_{\lambda i=j}, (a_i)_{\lambda i=j})$ for all $j \in \underline{n}$.

3.9. The natural transformation κ . We now define a natural transformation κ :

$$\begin{array}{ccc} \mathbf{S}(\tau \downarrow \tau \eta) & \xrightarrow{Sr} & \mathbf{S}C \\ Sq \downarrow & \Rightarrow \kappa & \downarrow \mathcal{X}_\tau \\ \mathbf{S}^2 C & & \\ \mu \downarrow & & \\ \mathbf{S}C & \xrightarrow{y_{\mathbf{S}C}} & \widehat{\mathbf{S}C} \end{array}$$

An object $(\lambda, (x_i)_{i \in \underline{m}}, (g_j)_{j \in \underline{n}}, (y_j)_{j \in \underline{n}})$ determines

$$(\lambda, (g_j)_{j \in \underline{n}}) \in (\mathcal{X}_\tau(y_j)_{j \in \underline{n}})(x_i)_{i \in \underline{m}} = \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i=j}, \tau y_j)$$

and thus by the Yoneda lemma,

$$\kappa(\lambda, (x_i)_{i \in \underline{m}}, (g_j)_{j \in \underline{n}}, (y_j)_{j \in \underline{n}}) : \widehat{\mathbf{S}C}(-, (x_i)_{i \in \underline{m}}) \longrightarrow \mathcal{X}_\tau(y_j)_{j \in \underline{n}}$$

which we take to be the corresponding component of κ . Naturality with respect to a morphism (7) follows from the fact that $(\mathcal{X}_\tau(\sigma, (a_i)_i))(x_i)_i(\lambda, (g_j)_j) = (\mathcal{X}_\tau(y'_j)_j)(\rho, (b_j)_j)(\lambda', (g'_j)_j)$, which itself follows from the conditions on the data comprising the morphism (7).

Lemma 3.10. *The natural transformation κ of Construction 3.9 exhibits \mathcal{X}_τ as a pointwise left Kan extension of $y_{\mathbf{S}C} \circ \mu \circ Sq$ along Sr .*

PROOF. We work pointwise, so postcompose κ with evaluation at an object $(x_i)_{i \in \underline{m}}$ of $\mathbf{S}C$. Furthermore, the functor $r : \tau \downarrow \tau \eta \rightarrow C$ is the right leg of a comma square and hence an opfibration, hence also Sr is an opfibration since \mathbf{S} preserves opfibrations. Thus we can also work pointwise in $(y_j)_{j \in \underline{n}}$, so altogether it suffices, for fixed $(x_i)_{i \in \underline{m}}$ and $(y_j)_{j \in \underline{n}}$, to show that

$$(8) \quad \begin{array}{ccccccc} Z_y & \xrightarrow{t_y} & \mathbf{S}(\tau \downarrow \tau \eta) & \xrightarrow{Sq} & \mathbf{S}^2 C & \xrightarrow{\mu} & \mathbf{S}C \\ \downarrow & \text{pb} & \downarrow Sr & & \Downarrow \kappa & & \downarrow y_{\mathbf{S}C} \\ 1 & \xrightarrow{(y_j)_j} & \mathbf{S}C & \xrightarrow{\mathcal{X}_\tau} & \widehat{\mathbf{S}C} & \xrightarrow{\text{ev}_{(x_i)_i}} & \mathbf{Set} \end{array}$$

$\begin{array}{c} \searrow \mathbf{S}C((x_i)_i, -) \\ = \end{array}$

is a colimit cocone. Here $Z_y = Z_{(y_j)_j}$ is temporary notation for the fibre of Sr over the object $(y_j)_{j \in \underline{n}}$, with inclusion functor $t_y : Z_y \rightarrow \mathbf{S}(\tau \downarrow \tau \eta)$. From the explicit description (3.8) of objects and arrows of $\mathbf{S}(\tau \downarrow \tau \eta)$, we see that the objects of Z_y are tuples $(\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}})$, where $\lambda : \underline{l} \rightarrow \underline{n}$ is monotone,

and for $j \in \underline{n}$, $g_j : \tau(z_i)_{\lambda i=j} \rightarrow y_j$. (Here we have needed new variables l and z_i , since m and x_i are already assigned at the beginning of the proof.) The morphisms of Z_y are of the form

$$(\sigma, (c_i)_{i \in \underline{l}}) : (\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}}) \longrightarrow (\lambda', (z'_i)_{i \in \underline{l}}, (g'_j)_{j \in \underline{n}})$$

where the permutation $\sigma \in \Sigma_l$ satisfies $\lambda'\sigma = \lambda$, and $c_i : z_i \rightarrow z'_{\sigma i}$ are maps in C such that $g'_j \tau(\sigma|_{\lambda i=j}, (c_i)_{\lambda i=j}) = g_j$ for all $j \in \underline{n}$.

We now unpack the cocone in (8) more explicitly. The composite $1 \rightarrow \mathbf{Set}$ along the bottom picks out

$$(\mathcal{X}_\tau(y_j)_j)(x_i)_i = \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i=j}, \tau y_j).$$

The composite functor $F := SC((x_i)_i, -) \circ \mu \circ Sq \circ t_y : Z_y \rightarrow \mathbf{Set}$ along the top of (8) is given on objects and arrows by

$$(\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}}) \mapsto SC((x_i)_{i \in \underline{m}}, (z_i)_{i \in \underline{l}}) \quad (\sigma, (c_i)_{i \in \underline{l}}) \mapsto (\sigma, (c_i)_i) \circ (-).$$

Note in particular that for $l \neq m$, $F(\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}})$ is the empty set.

Denoting by $\tilde{\kappa}$ the natural transformation in (8), we spell out its components: the component at $(\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}})$ is the function

$$\tilde{\kappa}_{(\lambda, (z_i)_{i \in \underline{l}}, (g_j)_{j \in \underline{n}})} : SC((x_i)_{i \in \underline{m}}, (z_i)_{i \in \underline{l}}) \longrightarrow \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i=j}, \tau y_j)$$

which when $l = m$, sends $(\sigma, (f_i)_{i \in \underline{m}}) : (x_i)_{i \in \underline{m}} \rightarrow (z_i)_{i \in \underline{l}}$ to $(\alpha, (k_j)_j)$, where $\alpha = \lambda\sigma$ and k_j is the composite

$$\tau(x_i)_{\alpha i=j} \xrightarrow{\tau(\sigma|_{\alpha i=j}, (f_i)_{\alpha i=j})} \tau(z_i)_{\lambda i=j} \xrightarrow{g_j} \tau y_j.$$

To show that $\tilde{\kappa}$ is a colimit cocone, we show that the associated functor from the category of elements,

$$\text{el}(F) \longrightarrow (\mathcal{X}_\tau(y_j)_j)(x_i)_i = \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\tau(x_i)_{\alpha i=j}, \tau y_j),$$

is surjective on objects and has connected fibres. Consider $(\alpha, (g_j)_j) \in (\mathcal{X}_\tau(y_j)_j)(x_i)_i$. Since in the bijective-monotone factorisation $\alpha = \lambda_\alpha \sigma_\alpha$ of α , the restrictions $\sigma_\alpha|_{\alpha i=j}$ of σ_α to the fibres of α are identities, each g_j is the composite

$$\tau(x_i)_{\alpha i=j} \xrightarrow{\tau(\sigma_\alpha|_{\alpha i=j}, (1_{x_i})_{\alpha i=j})} \tau(x_{\sigma_\alpha^{-1} i})_{\lambda_\alpha i=j} = \tau(x_i)_{\alpha i=j} \xrightarrow{g_j} \tau y_j$$

and so $(\alpha, (g_j)_j) = \tilde{\kappa}_{(\lambda_\alpha, (x_{\sigma_\alpha^{-1} i})_i, (g_j)_j)}(\sigma_\alpha, (1_{x_i})_i)$, whence $\tilde{\kappa}$ is jointly surjective. Consider now an arbitrary element in $\text{el}(F)$ in the fibre over $(\alpha, (g_j)_j)$. That's an element $((\lambda, (z_i)_i, (g'_j)_j), (\sigma, (f_i)_i))$ satisfying the equation

$$(9) \quad \tilde{\kappa}_{(\lambda, (z_i)_i, (g'_j)_j)}(\sigma, (f_i)_i) = (\alpha, (g_j)_j);$$

we must show it belongs to the same connected component of $\text{el}(F)$ as the object from before,

$$((\lambda_\alpha, (x_{\sigma_\alpha^{-1} i})_i, (g_j)_j), (\sigma_\alpha, (1_{x_i})_i)).$$

Equation (9) says that $\alpha = \lambda\sigma$ and g_j is the composite

$$\tau(x_i)_{\alpha i=j} \xrightarrow{\tau(\sigma|_{\alpha i=j}, (f_i)_{\alpha i=j})} \tau(z_i)_{\lambda i=j} \xrightarrow{g'_j} \tau y_j,$$

and so

$$(\sigma\sigma_\alpha^{-1}, (f_{\sigma_\alpha^{-1}i})_i) : (\lambda_\alpha, (x_{\sigma_\alpha^{-1}i})_i, (g_j)_j) \longrightarrow (\lambda, (z_i)_i, (g'_j)_j)$$

is a morphism of Z_y . This morphism lifts to a morphism in the category of elements since $(\sigma, (f_i)_i) = (\sigma\sigma_\alpha^{-1}, (f_{\sigma_\alpha^{-1}i})_i) \circ (\sigma_\alpha, (1_{x_i})_i)$ in SC . \square

Lemma 3.11. *In the situation of 3.7–3.9, $h_\tau : \mathcal{X}_\tau \rightarrow \mathcal{Y}_\tau$ is the unique natural transformation satisfying*

$$(10) \quad \begin{array}{c} S(\tau \downarrow \tau \eta) \xrightarrow{S_r} SC \\ Sq \downarrow \quad \Downarrow \kappa \quad \downarrow \\ S^2 C \xrightarrow{\quad} \mathcal{X}_\tau \xrightarrow{h_\tau} \mathcal{Y}_\tau \\ \mu \downarrow \quad \quad \quad \downarrow \\ SC \xrightarrow{y_{SC}} \widehat{SC} \end{array} = \begin{array}{c} S(\tau \downarrow \tau \eta) \xrightarrow{S_r} SC \\ Sq \downarrow \quad \Downarrow S_\gamma \quad \downarrow S(\tau \eta) \\ S^2 C \xrightarrow{S_\tau} SM \\ \mu \downarrow \quad \Downarrow \bar{\tau} \quad \downarrow \otimes \\ SC \xrightarrow{\tau} M \\ y_{SC} \searrow \quad \Downarrow \chi^\tau \quad \swarrow M(\tau, 1) \\ \widehat{SC} \end{array}$$

PROOF. Continuing the notation of the previous proof, and denoting by ψ the cocone

$$\begin{array}{ccccccc} Z_y & \xrightarrow{t_y} & S(\tau \downarrow \tau \eta) & \xrightarrow{Sq} & S^2 C & \xrightarrow{\mu} & SC \\ \downarrow & \text{pb} & \downarrow S_r & \Downarrow S_\gamma & \downarrow S_\tau & \Downarrow \bar{\tau} & \downarrow \tau \\ 1 & \xrightarrow{(y_j)_j} & SC & \xrightarrow{S(\tau \eta)} & SM & \xrightarrow{\otimes} & M \\ & & & & & & \downarrow M(\tau, 1) \\ & & & & & & \widehat{SC} \\ & & & & & & \downarrow \text{ev}_{(x_i)_i} \\ & & & & & & \mathbf{Set} \end{array}$$

for $(\lambda, (z_i)_i, (g_j)_j) \in Z_y$, we must verify that

$$(11) \quad h_{\tau, x, y} \tilde{\kappa}_{(\lambda, (z_i)_i, (g_j)_j)} = \psi_{(\lambda, (z_i)_i, (g_j)_j)}.$$

It is straightforward to unpack ψ to see that

$$\psi_{(\lambda, (z_i)_i, (g_j)_j)} : SC((x_i)_i, (z_i)_i) \longrightarrow M(\tau(x_i)_i, \otimes_j \tau y_j)$$

sends $(\sigma, (f_i)_i)$ to the composite

$$\tau(x_i)_{i \in \underline{m}} \xrightarrow{\tau(\sigma, (f_i)_i)} \tau(z_i)_{i \in \underline{m}} \xrightarrow{\bar{\tau}} \otimes_{j \in \underline{n}} \tau(z_i)_{\lambda i = j} \xrightarrow{\otimes_j g_j} \otimes_{j \in \underline{n}} \tau y_j.$$

We gave an explicit description of $\tilde{\kappa}_{(\lambda, (z_i)_i, (g_j)_j)}$ in the proof of Lemma 3.10, we know $h_{\tau, x, y}$ explicitly by definition, and so (11) is easily verified. \square

An immediate consequence of Lemma 3.11 and Lemma 2.2 is

Corollary 3.12. *A symmetric colax monoidal functor $\tau : SC \rightarrow M$ satisfies the hereditary condition of Definition 3.2 if and only if the composite square on the left*

$$\begin{array}{ccc} S(\tau \downarrow \tau \eta) & \xrightarrow{S_r} & SC \\ Sq \downarrow & \Downarrow S_\gamma & \downarrow S(\tau \eta) \\ S^2 C & \xrightarrow{S_\tau} & SM \\ \mu \downarrow & \Downarrow \bar{\tau} & \downarrow \otimes \\ SC & \xrightarrow{\tau} & M \end{array} \quad \begin{array}{ccc} \tau \downarrow \tau \eta & \xrightarrow{r} & C \\ q \downarrow & \Downarrow \gamma & \downarrow \tau \eta \\ SC & \xrightarrow{\tau} & M \end{array}$$

is exact, where γ is the comma square given on the right.

To apply this last result to the problem of showing that hereditariness implies exactness, we require one last general fact.

Lemma 3.13. *S preserves exact squares.*

PROOF. Given that the square on the left

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ p \downarrow & \xRightarrow{\phi} & \downarrow g \\ A & \xrightarrow{f} & C \end{array} \qquad \begin{array}{ccc} SP & \xrightarrow{Sq} & SB \\ Sp \downarrow & \xRightarrow{S\phi} & \downarrow Sg \\ SA & \xrightarrow{Sf} & SC \end{array}$$

is exact, we must verify the exactness of the square on the right, and this we do by verifying the explicit factorisation criterion of Lemma 2.4. To this end, we consider

$$(a_i)_{i \in \underline{n}} \in SA \quad (b_i)_{i \in \underline{n}} \in SB \quad (\rho, (\gamma_i)) : (fa_i)_i \rightarrow (gb_i)_i$$

and we must verify that the category $\mathcal{F} := \text{Fact}_{S\phi}((a_i)_i, (\rho, (\gamma_i)_i), (b_i)_i)$ is connected. By definition an object of \mathcal{F} is a factorisation of $(\rho, (\gamma_i)_i)$ of the form

$$(12) \quad (fa_i)_i \xrightarrow{(\rho_1, (f\alpha_i)_i)} (fpx_i)_i \xrightarrow{(1_n, (\phi_{x_i})_i)} (gqx_i)_i \xrightarrow{(\rho_2, (g\beta_i)_i)} (gb_i)_i.$$

To exhibit one such factorisation, using the exactness of ϕ one has a factorisation of γ_i as on the left

$$fa_i \xrightarrow{f\alpha_i} fpx_i \xrightarrow{\phi_{x_i}} gqx_i \xrightarrow{g\beta_i} gb_{\rho_i} \qquad (fa_i)_i \xrightarrow{(1_n, (f\alpha_i)_i)} (fpx_i)_i \xrightarrow{(1_n, (\phi_{x_i})_i)} (gqx_i)_i \xrightarrow{(\rho, (g\beta_i)_i)} (gb_i)_i$$

for each $i \in \underline{n}$, and these assemble together as on the right to give an object of \mathcal{F} . For a general object (12) of \mathcal{F} , the morphism $(\rho_1^{-1}, (1_{px_i})_i)$ exhibits it as being in the same component of an object in which $\rho_1 = 1_n$ and $\rho_2 = \rho$. Given two such special objects of \mathcal{F}

$$(fa_i)_i \xrightarrow{(1_n, (f\alpha_i)_i)} (fpx_i)_i \xrightarrow{(1_n, (\phi_{x_i})_i)} (gqx_i)_i \xrightarrow{(\rho, (g\beta_i)_i)} (gb_i)_i \quad (fa_i)_i \xrightarrow{(1_n, (f\alpha'_i)_i)} (fpx'_i)_i \xrightarrow{(1_n, (\phi_{x'_i})_i)} (gqx'_i)_i \xrightarrow{(\rho, (g\beta'_i)_i)} (gb_i)_i$$

one constructs an undirected path between them by constructing undirected paths for each i using the exactness of ϕ , adding in identities as necessary so that the constructed paths for different i are of the same length and orientation, and assembling these in the evident way to produce the required undirected path in \mathcal{F} . \square

Proposition 3.14. *Let $\tau : SC \rightarrow M$ be a symmetric colax monoidal functor. If τ is exact then it satisfies the hereditary condition.*

PROOF. $\overline{\tau}$ is an exact square by hypothesis, γ which defines $\tau \downarrow \tau \eta$ is exact since all comma squares are exact, and $S\gamma$ is an exact square since S preserves exact squares by Lemma 3.13. Thus by Corollary 3.12, τ satisfies the hereditary condition since exact squares compose vertically. \square

We combine the two main results of this section, Lemma 3.5 and Proposition 3.14, into the following:

Proposition 3.15. *An essentially surjective symmetric strong monoidal functor $\tau : SC \rightarrow M$ is exact if and only if it satisfies the hereditary condition.* \square

4. Kaufmann–Ward comma-category condition

4.1. Feynman categories. Kaufmann and Ward [18] define a *Feynman category* to be a symmetric strong monoidal functor $\tau : \mathbf{SC} \rightarrow M$ satisfying the three conditions

- (1) C is a groupoid
- (2) τ induces an equivalence of groupoids $\mathbf{SC} \simeq M_{\text{iso}}$
- (3) τ induces an equivalence of groupoids $\mathbf{S}(M \downarrow C)_{\text{iso}} \simeq (M \downarrow M)_{\text{iso}}$.

In this section we show that the comma-category condition (3), can be reformulated in terms of the hereditary condition. This is more or less implicit in [18]. This section does not seem to generalise to the enriched setting.

Proposition 4.2. *For an essentially surjective symmetric strong monoidal functor $\tau : \mathbf{SC} \rightarrow M$, the following are equivalent:*

- (1) $\tau : \mathbf{SC} \rightarrow M$ is hereditary and $\tau_{\text{iso}} : \mathbf{SC}_{\text{iso}} \rightarrow M_{\text{iso}}$ is an equivalence of groupoids
- (2) The natural map $w : \mathbf{S}(M \downarrow C)_{\text{iso}} \rightarrow (M \downarrow M)_{\text{iso}}$ is an equivalence of groupoids.

In view of Proposition 3.15, this shows:

Corollary 4.3. *A Feynman category is precisely a regular pattern $\tau : \mathbf{SC} \rightarrow M$ for which C is a groupoid and $\tau_{\text{iso}} : \mathbf{SC}_{\text{iso}} \rightarrow M_{\text{iso}}$ is an equivalence of groupoids.* \square

PROOF OF PROPOSITION 4.2. We immediately reduce to the strict situation, where τ is identity-on-objects. Lemma 4.4 below says that if just w is fully faithful, then also τ_{iso} is fully faithful (and therefore actually an isomorphism). So we can separate that out as a global assumption. Now w is full if and only if ε is injective by Lemma 4.6, and w is essentially surjective if and only if ε is surjective by Lemma 4.5. Finally, it is actually automatic that w is faithful (again by Lemma 4.4). \square

Lemma 4.4. *If $w : \mathbf{S}(M \downarrow C)_{\text{iso}} \rightarrow (M \downarrow M)_{\text{iso}}$ is full, respectively faithful, then $\tau_{\text{iso}} : \mathbf{SC}_{\text{iso}} \rightarrow M_{\text{iso}}$ is full, respectively faithful. Conversely, if τ_{iso} is faithful then w is faithful.*

PROOF. The first statements follow immediately from the commutative diagram

$$\begin{array}{ccc} \mathbf{SC}_{\text{iso}} & \xrightarrow{\tau_{\text{iso}}} & M_{\text{iso}} \\ \downarrow & & \downarrow \\ \mathbf{S}(M \downarrow C)_{\text{iso}} & \xrightarrow{w} & (M \downarrow M)_{\text{iso}} \end{array}$$

since the vertical maps are fully faithful.

For the last statement, the isomorphisms in the image of w are of the form

$$\begin{array}{ccc} \bigotimes_j \bigotimes_i x_i & \xrightarrow{a} & \bigotimes_j \bigotimes_i x'_i \\ \bigotimes_j g_j \downarrow & & \downarrow \bigotimes_j g'_j \\ \bigotimes_j y_j & \xrightarrow{b} & \bigotimes_j y'_j. \end{array}$$

If individually the horizontal isos can arise from \mathbf{SC} in at most one way, then taken together it is even harder to arise from $\mathbf{S}(M \downarrow C)$. \square

Lemma 4.5. *For a symmetric strict monoidal functor $\tau : \mathbf{SC} \rightarrow M$, such that $\mathbf{SC}_{\text{iso}} \simeq M_{\text{iso}}$, the following are equivalent:*

- (1) The map ε in the hereditary condition is surjective.

(2) *The natural map $w : \mathbf{S}(M \downarrow C)_{\text{iso}} \rightarrow (M \downarrow M)_{\text{iso}}$ is essentially surjective.*

PROOF. Throughout the proof we suppress τ on objects, since anyway τ is identity-on-objects.

We first prove that if ε is surjective, then w is essentially surjective. Given some object in $M \downarrow M$, that's precisely an element in the codomain of ε , say $f : \otimes_i x_i \rightarrow \otimes_j y_j$. By surjectivity of ε , there is an element $(\alpha, g_1, \dots, g_n)$ in the domain of ε , whose tensor product is f . Now just the sequence (g_1, \dots, g_n) is an object in $\mathbf{S}(M \downarrow C)$ and by assumption, their tensor product is isomorphic to f (by the permutation σ_α obtained from α).

Conversely, assuming w is essentially surjective, let us prove that ε is surjective. Given $f : \otimes_i x_i \rightarrow \otimes_j y_j$, an element in the codomain of ε , we need to construct an element on the left — that's a tuple $(\alpha, g_1, \dots, g_n)$ — such that the composite

$$\otimes_i x_i \xrightarrow{\sigma_\alpha} \otimes_j \otimes_k x_k \xrightarrow{\otimes_j g_j} \otimes_j y_j$$

is equal to f . (Here σ_α is the permutation part of α , obtained by permutation-monotone factorisation). Since w is essentially surjective, there exists an object $(\lambda', h'_1, \dots, h'_n)$ in $\mathbf{S}(M \downarrow C)$ (here $\lambda' : \underline{m} \rightarrow \underline{n}$ is monotone, and $h'_j : \otimes_{\lambda' i' = j'} x'_{i'} \rightarrow y'_{j'}$) whose tensor product is isomorphic to f :

$$\begin{array}{ccc} \otimes_i x_i & \xrightarrow[\sim]{a} & \otimes_{j'} \otimes_{\lambda' i' = j'} x'_{i'} \\ f \downarrow & & \downarrow \otimes_{j'} h'_{j'} \\ \otimes_j y_j & \xrightarrow[\sim]{b} & \otimes_{j'} y'_{j'}. \end{array}$$

Since τ_{iso} is full, both a and b come from \mathbf{SC} , and in particular can be written as a permutation followed by a tensor product of isos in C . Let ς be the permutation underlying a and let ρ be the permutation underlying b . Put

$$\alpha := \rho^{-1} \circ \lambda' \circ \varsigma.$$

The permutation $\rho : \underline{n} \xrightarrow{\sim} \underline{n}$ is such that with $j' = \rho j$ we have $y_j \simeq y'_{j'}$. Conjugating with this isomorphism we find

$$\begin{array}{ccc} \otimes_j \otimes_{\lambda' i' = \rho j} x'_{i'} & \xrightarrow[\sim]{\tilde{b}} & \otimes_{j'} \otimes_{\lambda' i' = j'} x'_{i'} \\ \otimes_j g'_j \downarrow & & \downarrow \otimes_{j'} h'_{j'} \\ \otimes_j y_j & \xrightarrow[\sim]{b} & \otimes_{j'} y'_{j'}. \end{array}$$

(The permutation $\tilde{\rho} : \underline{m} \xrightarrow{\sim} \underline{m}$ underlying \tilde{b} permutes the blocks according to the permutation b .) Except for some isos in C , the new map g'_j is essentially $h'_{\rho j}$. The tensor product of the new maps g'_j now have underlying indexing map $\lambda := \rho^{-1} \circ \lambda' \circ \tilde{\rho}$, which is different from λ' , but is still monotone. On the other hand, the permutation $\varsigma : \underline{m} \xrightarrow{\sim} \underline{m}$ is such that with $i' = \varsigma i$ we have $x_i \simeq x'_{i'}$. Using this, we can rewrite the upper left-hand corner

$$\otimes_j \otimes_{\lambda i' = \rho j} x'_{i'} \simeq \otimes_j \otimes_{\lambda \varsigma i = \rho j} x_i = \otimes_j \otimes_{\alpha i = j} x_i,$$

but a little care is needed with this substitution, since it may permute stuff inside each j -factor. However, this permutation can be absorbed into each g'_j and now called g_j (and this does not affect

λ), giving altogether

$$\begin{array}{ccccc}
 \bigotimes_i x_i & \xrightarrow{\sigma} & \bigotimes_j \bigotimes_{\alpha i=j} x_i & \longrightarrow & \bigotimes_{j'} \bigotimes_{\lambda' i'=j'} x'_{i'} \\
 f \downarrow & & \downarrow \bigotimes_j g_j & & \downarrow \bigotimes_{j'} h'_{j'} \\
 \bigotimes_j y_j & \xrightarrow{=} & \bigotimes_j y_j & \longrightarrow & \bigotimes_j y'_{j'}.
 \end{array}$$

The remaining permutation σ is monotone on λ -fibres by construction, and since $\alpha = \lambda \circ \sigma$, we see that $(\alpha, g_1, \dots, g_n)$ is a solution to our problem: by construction, ε applied to $(\alpha, g_1, \dots, g_n)$ is the original f . Hence ε is surjective. \square

Lemma 4.6. *For a symmetric strict monoidal functor $\tau : \mathbf{SC} \rightarrow M$, such that $\mathbf{SC}_{\text{iso}} \simeq M_{\text{iso}}$, the following are equivalent:*

- (1) *The map ε in the hereditary condition is injective.*
- (2) *The natural map $w : \mathbf{S}(M \downarrow C)_{\text{iso}} \rightarrow (M \downarrow M)_{\text{iso}}$ is full.*

PROOF. Throughout the proof we suppress τ on objects, since anyway τ is identity-on-objects.

Let us show that if w is full then ε is injective. Suppose we have two elements in the domain of ε both giving f . Say (α, g_j) and (α', g'_j) . (In both cases j runs to the same n : that's part of the data in f). This gives us now a commutative diagram of maps in M :

$$\begin{array}{ccccc}
 \bigotimes_j \bigotimes_i x_i & \xleftarrow{\sigma} & \bigotimes_i x_i & \xrightarrow{\sigma'} & \bigotimes_j \bigotimes_i x_i \\
 \bigotimes_j g_j \downarrow & & \downarrow f & & \downarrow \bigotimes_j g'_j \\
 \bigotimes_j y_j & \xlongequal{\quad} & \bigotimes_j y_j & \xlongequal{\quad} & \bigotimes_j y_j
 \end{array}$$

The outer vertical maps, $\bigotimes_j g_j$ and $\bigotimes_j g'_j$, are now two objects in $M \downarrow M$, exhibited isomorphic by means of $\sigma' \circ \sigma^{-1}$ and the identity at the bottom. Since w is full, this isomorphism comes from one in $\mathbf{S}(M \downarrow C)$, hence is given by a labelled permutation of \underline{n} . Since the bottom map is the identity, also the permutation $\sigma' \circ \sigma^{-1}$ is the identity on the outer tensor factors, those indexed by j . So σ and σ' agree on outer factors. But they are also monotone on fibres, so in fact they must agree completely. It follows that $g_j = g'_j$, and hence in particular also that $\alpha = \alpha'$.

Conversely, assuming that ε is injective, let us show that w is full. Given two objects in $M \downarrow M$ in the image of $\mathbf{S}(M \downarrow C)$, pictured vertically, and an iso between them (consisting of two isomorphisms, pictured horizontally):

$$\begin{array}{ccc}
 \bigotimes_j \bigotimes_i x_i & \xrightarrow{a} & \bigotimes_j \bigotimes_i x'_i \\
 \bigotimes_j g_j \downarrow & & \downarrow \bigotimes_j g'_j \\
 \bigotimes_j y_j & \xrightarrow{b} & \bigotimes_j y'_j
 \end{array}$$

A priori, we don't know that the j run to the same n , but in fact they do, because of the existence of b : since τ_{iso} is fully faithful, b is in fact the image of an isomorphism in \mathbf{SC} , which is to say that it is a labelled permutation. For the same reason, a is a labelled permutation too. Since the vertical maps are in the image of w , their corresponding α and α' have trivial permutation part. It follows that the permutation a must actually be a refinement of the permutation b . In particular we have necessarily $\alpha = \alpha'$, which is a monotone map, since it just comes from the tensor product.

It remains to check that a and b together actually form a valid isomorphism in $\mathbf{S}(M \downarrow C)$. We have found that the original square is the tensor product of a sequence of squares of the form

$$\begin{array}{ccc} \bigotimes_{i \in \alpha^{-1} \sigma^{-1} j} x_i & \xrightarrow{a} & \bigotimes_{i \in \alpha^{-1}(j)} x'_i \\ g_j \downarrow & & \downarrow g'_j \\ y_{\sigma^{-1} j} & \xrightarrow{b} & y'_j \end{array}$$

It remains to check that each of them commutes. For fixed $j \in \underline{n}$, the two composites in this individual square are both elements in the set

$$M\left(\bigotimes_{i \in \alpha^{-1} \sigma^{-1}(j)} x_i, y'_j\right),$$

so altogether they form a tuple which is an element in

$$\prod_{j \in \underline{n}} M\left(\bigotimes_{i \in \alpha^{-1} \sigma^{-1}(j)} x_i, y'_j\right).$$

That's in the $\sigma \circ \alpha$ summand of the domain of ε . If one of the squares did not commute, it would thus constitute two distinct elements in this set, with the same image under tensoring (the map ε). Since ε is injective, we conclude that in fact all those squares do commute, and hence form a valid isomorphism in $\mathbf{S}(M \downarrow C)$, as required. \square

5. Hermida-type adjunctions and the main theorem

5.1. Operads. By *operad* we always mean symmetric coloured operad (in **Set**), also known as symmetric multicategory. Hence, an operad P consists of a set I of objects (also called colours), for each pair $(a_1, \dots, a_n; b)$ consisting of a finite sequence of *input objects* (a_1, \dots, a_n) and one *output object* b , a set $P(a_1, \dots, a_n; b)$ of operations from (a_1, \dots, a_n) to b . Moreover there is an identity operation $1_a : (a) \rightarrow a$ in $P(a; a)$, and a substitution law satisfying the usual axioms.

We shall use various shorthand notation for sequences (a_1, \dots, a_n) , such as $(a_i)_{i \in \underline{n}}$, or just $(a_i)_i$, or even \mathbf{a} , when practical.

Operads form a 2-category **Opd**, the morphisms being morphisms of operads in the usual sense. A 2-cell

$$\begin{array}{ccc} & f & \\ P & \Downarrow \phi & Q \\ & g & \end{array}$$

consists of components $\phi_a : (fa) \rightarrow ga$, required to be natural with respect to all operations of P , that is, given $\alpha : (a_i)_i \rightarrow b$, one has $\phi_b f(\alpha) = g(\alpha)(\phi_{a_i})_i$.

A small category can be regarded as an operad with only unary operations, and in this way **Cat** becomes a coreflective sub-2-category of **Opd** (the coreflector picks out the unary part of an operad.) Various notions from category theory makes sense also for operads: in particular, a morphism of operads $f : P \rightarrow Q$ is called *fully faithful* if it induces bijections on multihomsets $P(\mathbf{a}; b) \xrightarrow{\sim} Q(f\mathbf{a}; f(b))$. Gabriel factorisation works the same for operads as for categories: every operad morphism factors as bijective-on-objects followed by fully faithful. Just as in the category case, this is an enhanced factorisation system. And just as in the category case, one can always choose the bijective-on-objects part to be actually identity-on-objects (with which choice the factorisation is actually unique).

5.2. Hermida adjunction. The interaction between symmetric monoidal categories and operads is encapsulated by the adjunction

$$\mathbf{SMC} \begin{array}{c} \xleftarrow{\mathbf{F}} \\ \xrightarrow[\text{End}]{\perp} \end{array} \mathbf{Opd}$$

which we shall call the *Hermida adjunction* in honour of Claudio Hermida, who studied this adjunction in the non-symmetric case [16] (see also [13], Theorem 4.2). The functor End is a basic ingredient in operad theory, and applies also to symmetric monoidal categories M that aren't necessarily strict: the *endomorphism operad* $\text{End}(M)$ has objects those of M , and sets of operations given by

$$\text{End}(M)((x_1, \dots, x_n); y) = M(\bigotimes_{i=1}^n x_i, y).$$

A P -algebra in M is the same thing as a morphism of operads $P \rightarrow \text{End}(M)$.

The left adjoint \mathbf{F} can be obtained rather formally from the 2-monad \mathbf{S} (see [29] Corollary 6.4.7). Here we content ourselves to give an explicit description. An object of $\mathbf{F}P$ is a finite sequence of objects in P . A morphism $(x_1, \dots, x_m) \rightarrow (y_1, \dots, y_n)$ in $\mathbf{F}P$, consists of an *indexing function*

$$\alpha : \underline{m} \rightarrow \underline{n}$$

together with for each $j \in \underline{n}$, an operation $\phi_j : (x_i)_{\alpha i=j} \rightarrow y_j$. The category structure of $\mathbf{F}P$ comes from substitution in P and the composition of (indexing) functions. Thus the homs of $\mathbf{F}P$ are given by

$$\mathbf{F}P((x_i)_{i \in \underline{m}}, (y_j)_{j \in \underline{n}}) = \sum_{\alpha : \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} P((x_i)_{\alpha i=j}; y_j).$$

Note that for a category C regarded as an operad, we have $\mathbf{F}C = \mathbf{S}C$.

The counit component at $M \in \mathbf{SMC}$

$$\varepsilon_M^H : \mathbf{F}(\text{End}(M)) \longrightarrow M \quad (x_i)_i \mapsto \bigotimes_i x_i$$

has object map as indicated, and hom functions

$$\mathbf{F}(\text{End}(M))((x_i)_{i \in \underline{m}}, (y_j)_{j \in \underline{n}}) \longrightarrow M(\bigotimes_{i \in \underline{m}} x_i, \bigotimes_{j \in \underline{n}} y_j)$$

which send

$$(\alpha : \underline{m} \rightarrow \underline{n}, (g_j : \bigotimes_{\alpha i=j} x_i \rightarrow y_j)_j)$$

to the composite

$$\bigotimes_{i \in \underline{m}} x_i \xrightarrow{\sigma} \bigotimes_{i \in \underline{m}} x_{\sigma^{-1}i} = \bigotimes_{j \in \underline{n}} \bigotimes_{\alpha i=j} x_i \xrightarrow{\bigotimes_j g_j} \bigotimes_{j \in \underline{n}} y_j$$

where $\sigma = \sigma_\alpha$ is the permutation part of α (given by the permutation/monotone factorisation).

The component of the unit at $P \in \mathbf{Opd}$ is given on objects as

$$\begin{aligned} \eta_P^H : P &\longrightarrow \text{End}(\mathbf{F}P) \\ x &\longmapsto (x). \end{aligned}$$

More interesting is to see what η^H does on sets of operations: it is a map

$$P(x_1, \dots, x_m; y) \rightarrow \text{End}(\mathbf{F}P)((x_1), \dots, (x_m); (y))$$

but the last set we can unravel as

$$= \mathbf{F}P((x_1, \dots, x_m), (y))$$

since the tensor product in FP is just concatenation of sequences;

$$= \sum_{\alpha: \underline{m} \rightarrow \underline{1}} \prod_{j \in \underline{1}} P((x_i)_{\alpha i=j}; y_j)$$

by definition of hom sets in FP ;

$$= P(x_1, \dots, x_m; y)$$

since there exists only one indexing map $\underline{m} \rightarrow \underline{1}$. In the end, η_P^H is the identity map on operations. In conclusion:

Lemma 5.3. *The components of η^H , the unit for the Hermida adjunction, are fully faithful operad maps.* \square

5.4. Pinned operads. We use the term *pinned* for an object (e.g. a symmetric monoidal category, an operad, or a substitute) equipped with a map singling out some objects (the pins).

A *pinned operad* is an operad P equipped with a morphism of operads $\phi : C \rightarrow P$, where C is a category regarded as an operad with only unary operations. Pinned operads assemble into a 2-category

$$pOpd := Cat \downarrow Opd.$$

The 2-category $pSMC$ of pinned symmetric strict monoidal categories was described above in 1.6.

5.5. Pinned Hermida adjunction. We now construct the pinned analogue of the Hermida adjunction. Its construction is a formal consequence of the fact that we have a commutative triangle

$$\begin{array}{ccc} SMC & \xleftarrow{F} & Opd \\ & \nwarrow S \quad \nearrow & \\ & Cat & \end{array}$$

With this, the Hermida adjunction $F \dashv \text{End}$ induces formally an adjunction between the comma categories:

$$pSMC \begin{array}{c} \xleftarrow{F_p} \\ \perp \\ \xrightarrow{\text{End}_p} \end{array} pOpd.$$

Let us spell out the ingredients. The object maps of End_p and F_p are

$$SC \xrightarrow{\tau} M \quad \mapsto \quad C \xrightarrow{\eta_C^H} \text{End}(SC) \xrightarrow{\text{End}(\tau)} \text{End}(M) \qquad C \xrightarrow{\phi} P \quad \mapsto \quad SC \xrightarrow{F(\phi)} FP$$

respectively, and since End and F are 2-functors and η^H is 2-natural, these definitions extend in the obvious way to arrows and 2-cells.

The component of the unit of $F_p \dashv \text{End}_p$ at $\phi : C \rightarrow P$ is the square

$$(13) \quad \begin{array}{ccc} C & \xrightarrow{\phi} & P \\ \parallel & & \downarrow \eta_P^H \\ C & \xrightarrow{\eta_C^H} \text{End}(SC) \xrightarrow{\text{End}(F(\phi))} & \text{End}(FP) \end{array}$$

and the component of the counit of $F_p \dashv \text{End}_p$ at $\tau : SC \rightarrow M$ is the square

$$\begin{array}{ccc} SC & \xrightarrow{F(\eta_C^H)} & F(\text{End}(SC)) \xrightarrow{F(\text{End}(\tau))} F(\text{End}(M)) \\ \parallel & & \downarrow \varepsilon_M^H \\ SC & \xrightarrow{\tau} & M \end{array}$$

This last diagram commutes by the naturality of ε^H and the triangle equations for the adjunction $F \dashv \text{End}$.

5.6. Substitudes [11]. The notion of substitute was introduced by Day and Street [11] as a general setting for substitution. It is a common generalisation of operad and (symmetric) monoidal category. See the appendix of [4] for a concise account of the basic theory. A substitute is like an operad, but allowing for a category of objects instead of just a set of objects. The data is

- a category C
- a functor $P : (SC)^{\text{op}} \times C \rightarrow \mathbf{Set}$
- composition and unit laws, subject to non-surprising axioms.

For the present purposes, the most convenient is to package the definition into the following (cf. [12, 6.3] for the equivalence between the two formulations of the definition): A *substitute* is a pinned operad $\phi : C \rightarrow P$ in which ϕ is the identity on objects. We denote by **Subst** the full sub-2-category of **pOpd** consisting of the substitutes, and denote by $J : \mathbf{Subst} \rightarrow \mathbf{pOpd}$ the inclusion.

5.7. Substitute coreflection. Since the bijective-on-objects morphisms form the left-hand class of an orthogonal factorisation system (Gabriel factorisation), by general principles, the inclusion functor $J : \mathbf{Subst} \rightarrow \mathbf{pOpd}$ has a right adjoint, denoted $(-)'$, forming an adjunction

$$\mathbf{pOpd} \begin{array}{c} \xleftarrow{J} \\ \perp \\ \xrightarrow{(-)'} \end{array} \mathbf{Subst}$$

Explicitly, given a pinned operad $\phi : C \rightarrow P$, factoring it as identity-on-objects then fully faithful

$$\begin{array}{ccc} C & \xrightarrow{\phi} & P \\ & \searrow \phi' & \nearrow \varepsilon_\phi \\ & P' & \end{array}$$

one obtains a substitute $\phi' : C \rightarrow P'$, and at the same time the ϕ -component of the counit $J \circ (-)' \Rightarrow \text{id}_{\mathbf{pOpd}}$.

It is suggestive to write the operad part of $\phi' : C \rightarrow P'$ as $P|C$: it is the operad base-changed to its pins.

$$\begin{array}{ccc} C & \xrightarrow{\phi} & P \\ & \searrow \phi' & \nearrow \\ & P|C & \end{array}$$

The operad $P|C$ has the same objects as C by construction, and operations

$$P|C(x_1, \dots, x_m; y) = P(\phi x_1, \dots, \phi x_m; \phi y).$$

5.8. The substitute adjunction. Now compose the pinned Hermida adjunction 5.5 with the coreflection of substitutes into pinned operads 5.7:

$$pSMC \begin{array}{c} \xleftarrow{F_p} \\ \perp \\ \xrightarrow{\text{End}_p} \end{array} pOpd \begin{array}{c} \xleftarrow{J} \\ \perp \\ \xrightarrow{(-)'} \end{array} \mathbf{Subst}$$

The composed adjunction

$$(14) \quad pSMC \begin{array}{c} \xleftarrow{F_{iop}} \\ \perp \\ \xrightarrow{\text{End}_{iop}} \end{array} \mathbf{Subst}$$

goes like this: the left adjoint takes a substitute $C \rightarrow P$ to the pinned monoidal category $SC \rightarrow FP$, and the right adjoint takes a pinned monoidal category $SC \rightarrow M$ to $C \rightarrow \text{End}(M)|C$.

Unlike the Hermida adjunction and the pinned version, this one has invertible unit:

Proposition 5.9. *The unit for the substitute adjunction $F_{iop} \dashv \text{End}_{iop}$ is invertible.*

PROOF. Let $\phi : C \rightarrow P$ be a substitute. In the diagram

$$\begin{array}{ccccc} C & & \xrightarrow{\phi} & & P \\ & \searrow \phi & & \nearrow & \\ & & P & & \\ & & \vdots & & \downarrow \eta_P^H \\ C & \xrightarrow{\eta_C^H} & \text{End}(SC) & \xrightarrow{\text{End}(F(\phi))} & \text{End}(F(P)) \\ & \searrow \text{io} & & \nearrow \text{ff} & \\ & & \text{End}(FP)|C & & \end{array}$$

the back rectangle is the unit η_ϕ^p (cf. (13)). The unit we are after, η_ϕ^{iop} , is obtained by Gabriel factorising its horizontal arrows: this induces the dotted arrow by functoriality of the factorisation, and the left-hand square is now η_ϕ^{iop} . It is clearly invertible if and only if the dotted arrow is invertible. But the dotted arrow is invertible because it compares two Gabriel factorisations of $\text{End}(F(\phi)) \circ \eta_C^H$: on one hand ϕ followed by η_P^H (the latter being fully faithful by Lemma 5.3) and on the other hand the factorisation at the bottom of the diagram. \square

5.10. Counit of the substitute adjunction. Thanks to Proposition 5.9 the 2-category \mathbf{Subst} embeds as a full coreflective sub-2-category of $pSMC$. In the remainder of this section we determine for which pinned symmetric strict monoidal categories $\tau : SC \rightarrow M$, the counit ε_τ^{iop} is invertible, and so come to a characterisation of the substitutes amongst pinned symmetric monoidal categories. Fundamental to this characterisation is the hereditary condition 3.2.

We begin the unpacking: first we describe the substitute $\text{End}_{iop}(\tau)$. It is obtained by Gabriel factorising the pinned operad $\text{End}_p(\tau)$, which is the top composite here:

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C^H} & \text{End}(SC) & \xrightarrow{\text{End}(\tau)} & \text{End}(M) \\ & \searrow & & \nearrow \epsilon & \\ & & \text{End}_{iop}(\tau) & & \\ & & \text{End}(M)|C & & \end{array}$$

Note that the operad $\text{End}(M)|C$ has objects those of C , and operation sets

$$\text{End}(M)|C(x_1, \dots, x_m; y) = \text{End}(M)(\tau x_1, \dots, \tau x_m; \tau y) = M(\tau x_1 \otimes \dots \otimes \tau x_m, \tau y).$$

Now apply F_{iop} , rendered as the first row in the next diagram; the diagram as a whole is the counit $\varepsilon_\tau^{\text{iop}}$, the key part being of course the right-hand composite:

$$(15) \quad \begin{array}{ccc} SC & \xrightarrow{F_{\text{iop}}(\text{End}_{\text{iop}}(\tau))} & F(\text{End}(M)|C) \\ \parallel & & \downarrow F(\epsilon) \\ SC & \xrightarrow{\tau} & M \\ & & \downarrow \varepsilon_M^H \\ & & F(\text{End}(M)) \end{array}$$

The objects of the symmetric monoidal category $F(\text{End}(M)|C)$ are sequences of objects in C , and the counit $\varepsilon_\tau^{\text{iop}}$ sends (x_1, \dots, x_m) to $\tau x_1 \otimes \dots \otimes \tau x_m$. The hom sets of $F(\text{End}(M)|C)$ are

$$F(\text{End}(M)|C)((x_i)_{i \in \underline{m}}, (y_j)_{j \in \underline{n}}) = \sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} M(\bigotimes_{\alpha i = j} \tau x_i, \tau y_j),$$

and the counit map to $M(\bigotimes_i \tau x_i, \bigotimes_j \tau y_j)$ is ε_M^H : it sends

$$(\alpha, g_1, \dots, g_n)$$

to the composite

$$\bigotimes_{i \in \underline{m}} \tau x_i \xrightarrow{\sigma} \bigotimes_{i \in \underline{m}} \tau x_{\sigma^{-1}i} = \bigotimes_{j \in \underline{n}} \bigotimes_{\alpha i = j} \tau x_i \xrightarrow{\bigotimes_j g_j} \bigotimes_{j \in \underline{n}} \tau y_j.$$

This is precisely the map in the hereditary condition! In conclusion:

Lemma 5.11. *For a pinned symmetric monoidal category $\tau : SC \rightarrow M$, the map in the hereditary condition 3.2 is precisely (the arrow part of) the counit component $\varepsilon_\tau^{\text{iop}}$ of the substitute adjunction.* \square

Proposition 5.12. *A pinned symmetric monoidal category $\tau : SC \rightarrow M$ is in the essential image of F_{iop} if and only if τ is bijective on objects and satisfies the hereditary condition 3.2.*

PROOF. The essential image of F_{iop} is characterised as where $\varepsilon_\tau^{\text{iop}}$ is invertible. This happens when the functor on the right in (15) is bijective on objects and fully faithful. Fully faithful is equivalent to the hereditary condition by Lemma 5.11, and bijective on objects is equivalent to τ being bijective on objects, as clearly $F_{\text{iop}}(\text{End}_{\text{iop}}(\tau))$ is always so by construction. \square

Since bijective on objects plus the hereditary condition implies exactness, we have arrived at the first part of the main theorem:

Theorem 5.13. *The essential image of the fully faithful functor $\mathbf{Subst} \rightarrow \mathbf{pSMC}$ is precisely the category of (strictified) regular patterns.* \square

5.14. Operads. There are three canonical embeddings of the category of operads into substitutes. For a given operad P , the options are: the trivial pinning, where $C = \text{obj}(P)$, the discrete category of objects; the canonical groupoid pinning, where $C = P_1^{\text{iso}}$, the groupoid of invertible unary operations; and the full pinning, where $C = P_1$, the category of all unary operations.

We are interested at the moment in the canonical groupoid pinning,

$$P_1^{\text{iso}} \rightarrow P.$$

It defines a fully faithful functor

$$\mathbf{Opd} \rightarrow \mathbf{Subst}.$$

We can now characterise the image of the composite fully faithful functor

$$\mathbf{Opd} \rightarrow \mathbf{Subst} \rightarrow \mathbf{pSMC},$$

which is the second part of our main theorem:

Theorem 5.15. *The essential image of the fully faithful functor $\mathbf{Opd} \rightarrow \mathbf{Subst} \rightarrow \mathbf{pSMC}$ consists precisely of the Feynman categories in the sense of Kaufmann and Ward.*

PROOF. First we check that a pinned symmetric monoidal category in the image is a Feynman category, using 4.3. We already know that it satisfies the hereditary condition, and by construction P_1^{iso} is a groupoid. It remains to check that

$$SP_1^{\text{iso}} \rightarrow (\mathbf{FP})_{\text{iso}}$$

is an equivalence. These two categories have the same objects, namely sequences of objects in P . The arrows in \mathbf{FP} from \mathbf{x} to \mathbf{y} form the hom set

$$\sum_{\alpha: \underline{m} \rightarrow \underline{n}} \prod_{j \in \underline{n}} P((x_i)_{i \in \alpha^{-1}(j)}; y_j)$$

They are composed as operations in P . There is an obvious forgetful functor to \mathbf{Set} , given by returning the indexing set for the sequence. For an arrow to be invertible, at least its underlying α must be a bijection, and furthermore it is then clear that the involved operations have to be invertible unary operations.

Conversely, if we start with a Feynman category $\tau : \mathbf{SC} \rightarrow M$, the image in \mathbf{Subst} is the substitute $C \rightarrow \text{End}(M)|C$, and since C is a groupoid and $\mathbf{SC} \simeq M_{\text{iso}}$, it is clear that the invertible unary operations of $\text{End}(M)|C$ are precisely those of C , which is the condition for the substitute to be in the image of the groupoid-pinning functor $\mathbf{Opd} \rightarrow \mathbf{Subst}$. \square

5.16. Two other subcategories of \mathbf{RPat} equivalent to the category of operads. Let us note that there are two other subcategories of \mathbf{RPat} which are equivalent to the category of operads, given by the two other natural embeddings of \mathbf{Opd} into \mathbf{Subst} : one takes an operad P to its discrete pinning $\text{obj}(P) \rightarrow P$, and the other takes P to its full pinning $P_1 \rightarrow P$.

The first corresponds to the subcategory of discrete substitutes, which in turn corresponds to the subcategory of regular patterns $\mathbf{SC} \rightarrow M$ for which C is discrete. While this is obviously a stronger condition than the first Feynman-category condition, that of being a groupoid, on the other hand the second Feynman-category condition, $\mathbf{SC} \simeq M_{\text{iso}}$ is not in general satisfied. Imposing this second condition on top of the discreteness condition gives the notion of discrete Feynman category, mentioned in [18] to be related to operads. More precisely this notion corresponds to operads in which there are no unary operations other than the identities (the locus of operads for which the groupoid-pinning and discrete-pinning embeddings coincide).

The second embedding of operads into substitutes, endowing an operad P with its full pinning $P_1 \rightarrow P$, has as essential image that of *normal* substitutes, namely those whose unary operations

are precisely those coming from C . These can also be characterised as those for which the square constituting the unique substitute morphism to the terminal substitute $1 \rightarrow \mathbf{Comm}$ is a pullback:

$$\begin{array}{ccc} C & \longrightarrow & P \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \mathbf{Comm}. \end{array}$$

The corresponding subcategory in **RPat** is the full subcategory spanned by the regular patterns $SC \rightarrow M$ which are pullbacks of the terminal regular pattern $S1 \rightarrow \mathbf{FinSet}$. In detail, for any (strictified) regular pattern, which in virtue of Theorem 5.13 is of the form $SC \rightarrow FP$, we have a commutative square of symmetric strong monoidal functors

$$\begin{array}{ccc} SC & \longrightarrow & FP \\ \downarrow & & \downarrow \\ S1 & \longrightarrow & \mathbf{FinSet}, \end{array}$$

which clearly is a pullback precisely when the previous square is.

5.17. Algebras for substitutes. Given a substitute $\phi : C \rightarrow P$, and a symmetric monoidal category W an algebra consists of

- a functor $A : C \rightarrow W$
- for each $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in C$, an action map

$$P(\mathbf{x}; y) \longrightarrow W(A(x_1) \otimes \cdots \otimes A(x_n), A(y))$$

satisfying some conditions. One of these conditions says that for every arrow $f : x \rightarrow y$ in C , the action of the unary operation $\phi(f)$ is equal to $A(f) : A(x) \rightarrow A(y)$. The remaining conditions are those of an algebra for the operad P ; in fact, the first condition implies that functoriality in C follows from the P -algebra axioms, so to give an algebra for $\phi : C \rightarrow P$ in W amounts just to give an algebra for the operad P in W .

Just as in the case of operads, the notion of algebra can also be described in terms of a substitute of endomorphisms: the *endomorphism substitute* E^A of a functor $A : C \rightarrow W$ is given in elementary terms as the substitute with category C and $E^A(x_1, \dots, x_n; y) = W(A(x_1) \otimes \cdots \otimes A(x_n), A(y))$. An algebra structure on $A : C \rightarrow W$ is now the same thing as an identity-on- C substitute map $\phi \rightarrow E^A$, that is, an operad map $P \rightarrow E^A$ under C . The endomorphism substitute can be described more conceptually as

$$C \rightarrow \mathbf{End}(W)|C,$$

so altogether the notion of algebra is conveniently formulated in terms of the substitute adjunction: the endomorphism substitute of $A : C \rightarrow W$ is precisely $\mathbf{End}_{\text{iop}}(\tau^A)$ where $\tau^A : SC \rightarrow W$ is the tautological factorisation of A through SC , and an algebra is an identity-on- C substitute map $\phi \rightarrow \mathbf{End}_{\text{iop}}(\tau^A)$. By adjunction, this is the same thing as an identity-on- C pinned symmetric monoidal functor

$$\mathbf{F}_{\text{iop}}(\phi) \rightarrow \tau^A.$$

Giving this amounts just to giving $\alpha : FP \rightarrow W$ (a symmetric monoidal functor), that is, an algebra for the regular pattern corresponding to ϕ .

This works also for algebra homomorphisms: a homomorphism of substitute algebras is a natural transformation $u : A \Rightarrow B$ compatible with the action maps. In terms of endomorphism substitutes,

this compatibility amounts to commutativity of the diagram (of operads under C)

$$(16) \quad \begin{array}{ccc} & E^A & \\ P \nearrow & & \searrow u(-,*) \\ & E^B & \nearrow u(*,-) \\ & & E^{A,B} \end{array}$$

where $E^{A,B}$ is the W -bimodule $SC^{\text{op}} \times C \rightarrow W$ given by $E^{A,B}(\mathbf{x}; y) = W(A(x_1) \otimes \cdots \otimes A(x_n), B(y))$, $u(-, *)$ is induced by postcomposition with u_y , and $u(*, -)$ is induced by precomposition with $u_{x_1} \otimes \cdots \otimes u_{x_n}$. Now the natural transformation $u : A \Rightarrow B$ extends tautologically to a monoidal natural transformation $\tau^u : \tau^A \Rightarrow \tau^B$, which in turn extends to a monoidal natural transformation

$$\mathbf{FP} \begin{array}{c} \xrightarrow{\alpha} \\ \downarrow \\ \xrightarrow{\beta} \end{array} W,$$

which is the corresponding homomorphism of regular-pattern algebras. The components of this monoidal natural transformation are the same as those of τ^u , since SC and FP have the same object set; naturality in arrows in FP is a consequence of (16), and monoidality follows from construction.

Clearly this construction can be reversed, to construct an substitute-algebra homomorphism from a monoidal natural transformation. Altogether we obtain the following proposition.

Proposition 5.18. *Algebras for a substitute are the same thing as algebras for the corresponding regular pattern. More precisely for each symmetric monoidal category W , there is an isomorphism of categories between the category of algebras in W for a substitute, and the category of algebras in W for the corresponding regular pattern.* \square

Since algebras for a substitute $\phi : C \rightarrow P$ are just algebras for the operad P , we immediately get also, via the embedding $\mathbf{Opd} \rightarrow \mathbf{Subst}$ by canonical groupoid pinning:

Corollary 5.19. *Algebras for an operad are the same thing as algebras for the corresponding Feynman category.* \square

Appendix A. Power coherence

In this appendix we recall coherence for symmetric monoidal categories, from the point of view of Power's general approach to coherence [25]. This point of view takes as input the 2-monad \mathbf{S} on \mathbf{Cat} for symmetric monoidal categories, and produces the coherence theorem for symmetric monoidal categories. The formulation of this result given in Lemma A.4 below, is most convenient for us for the purposes of studying regular patterns and Feynman categories.

A.1. 2-monads and their algebras. A 2-monad T on a 2-category \mathcal{K} is just a monad in the sense of \mathbf{Cat} -enriched category theory. Thus one has the usual data of a monad

$$T : \mathcal{K} \longrightarrow \mathcal{K} \quad \eta : 1_{\mathcal{K}} \longrightarrow T \quad \mu : T^2 \longrightarrow T$$

but where T is a 2-functor, and η and μ are 2-natural, and the axioms are written down exactly as before. The extra feature is that now one has several different types of algebras, and several different types of algebra morphisms, and thus a variety of alternative 2-categories of algebras. For instance

a *pseudo T -algebra* structure on $A \in \mathcal{K}$ consists of the data of an action $a : TA \rightarrow A$, as well as invertible coherence 2-cells $\bar{a}_0 : 1_A \rightarrow a\eta_A$ and $\bar{a}_2 : aT(a) \rightarrow a\mu_A$, which satisfy:

$$\begin{array}{ccccc}
 a & \xrightarrow{\bar{a}_0 a} & a\eta_A a & & aT(a)T^2(a) \xrightarrow{\bar{a}_2 T^2(a)} a\mu_A T^2(a) & & aT(a)T(\eta_A) \xleftarrow{aT(\bar{a}_0)} a \\
 & \searrow \text{id} & \downarrow \bar{a}_2 \eta_{TA} & & \downarrow aT(\bar{a}_2) & = & \downarrow \bar{a}_2 \mu_{TA} \\
 & & a & & aT(a)T(\mu_A) \xrightarrow{\bar{a}_2 T(\mu_A)} a\mu_A \mu_{TA} & & a \xleftarrow{\text{id}} a
 \end{array}$$

When these coherence isomorphisms are identities, we have a *strict T -algebra* on A . There are also algebra types in which the coherence 2-cells are non-invertible, but these are not important for us here.

A *lax morphism* $(A, a) \rightarrow (B, b)$ between pseudo T -algebras is a pair (f, \bar{f}) , where $f : A \rightarrow B$ and $\bar{f} : bT(f) \rightarrow fa$, satisfying the following axioms:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & f & \\
 \bar{b}_0 f \swarrow & & \searrow f \bar{a}_0 \\
 bT(f)\eta_A & \xrightarrow{\bar{f}\eta_A} & fa\eta_A
 \end{array} & = & \begin{array}{ccc}
 bT(b)T^2(f) & \xrightarrow{\bar{b}_2 T^2(f)} & b\mu_B T^2(f) \\
 bT(\bar{f}) \downarrow & & \downarrow \bar{f}\mu_A \\
 bT(fa) & & fa\mu_A \\
 \bar{f}T(a) \searrow & & \nearrow f\bar{a}_2 \\
 & faT(a) &
 \end{array}
 \end{array}$$

When \bar{f} is an isomorphism, f is said to be a *pseudo morphism*, and when \bar{f} is an identity, f is said to be *strict*. Given lax T -algebra morphisms f and $g : (A, a) \rightarrow (B, b)$, a T -algebra 2-cell $f \rightarrow g$ is a 2-cell $\phi : f \rightarrow g$ in \mathcal{K} such that $\bar{g}(bT(\phi)) = (\phi a)\bar{f}$. In the case $T = \mathbf{S}$ one has

- lax morphism = symmetric lax monoidal functor,
- pseudo morphism = symmetric strong monoidal functor,
- strict morphism = symmetric strict monoidal functor, and
- algebra 2-cell = monoidal natural transformation.

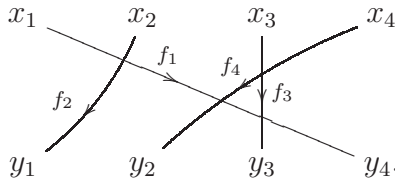
The standard notations for some of the various 2-categories of algebras of a 2-monad T are

- $\text{Ps-}T\text{-Alg}$: objects are pseudo T -algebras, morphisms are pseudo morphisms,
- $T\text{-Alg}_s$: objects and morphisms are strict, and
- $T\text{-Alg}_l$: objects are strict and morphisms are lax,

A.2. The free symmetric-monoidal-category 2-monad. The 2-monad \mathbf{S} is described explicitly in Section 5.1 of [28]. For a category C , an object of $\mathbf{S}C$ is a finite sequence of objects of C , and a morphism is of the form

$$(\rho, (f_i)_{1 \leq i \leq n}) : (x_i)_{1 \leq i \leq n} \longrightarrow (y_i)_{1 \leq i \leq n}$$

where $\rho \in \Sigma_n$ is a permutation, and for $i \in \underline{n} = \{1, \dots, n\}$, $f_i : x_i \rightarrow y_{\rho i}$. Intuitively such a morphism is a permutation labelled by the arrows of C , as in



The unit $\eta_C : C \rightarrow \mathbf{S}C$ is given by the inclusion of sequences of objects of length 1. The multiplication $\mu_C : \mathbf{S}^2 C \rightarrow \mathbf{S}C$ is given on objects by concatenation, and on arrows by the substitution of labelled

permutations. Given a symmetric monoidal category M , $(X_i)_i \mapsto \bigotimes_i X_i$ is the effect on objects of a functor $\bigotimes : \mathbf{S}M \rightarrow M$, whose arrow map is described using the symmetries of M . This is the action of a pseudo \mathbf{S} -algebra structure on M . The unit coherences (those denoted \bar{a}_0 in the general definition) in this case are identities, and the components of \bar{a}_2 come from the associators and unit coherences of M . The strictness of M as a symmetric monoidal category, is the same thing as its strictness as an \mathbf{S} -algebra. In fact, aside from the strictness of the unit coherences, and the specification of the tensor product as an n -ary tensor product for all n , rather than just the cases $n = 0$ (the unit usually written as I) and $n = 2$ (the usual binary tensor product $(A, B) \mapsto A \otimes B$), a pseudo \mathbf{S} -algebra is *exactly* a symmetric monoidal category in the usual sense.

The coherence theorem for symmetric monoidal categories says that every symmetric monoidal category is equivalent to a strict one. Mac Lane's original proof of this involved a detailed combinatorial analysis. However from the point of view of Power's general approach [25], this result comes out of how \mathbf{S} interacts with the *Gabriel factorisation system* on \mathbf{Cat} , by which every functor $f : A \rightarrow B$ is factored as

$$A \xrightarrow{g} C \xrightarrow{h} B$$

where g is bijective on objects, and h is fully faithful.

A.3. The Gabriel factorisation system. The Gabriel factorisation system, in which the left class is that of bijective-on-objects functors, and the right class consists of the fully faithful functors, is an orthogonal factorisation system, meaning that arrows in the left class admit a unique lifting property with respect to arrows in the right class. One way to formulate this, is to say that for any bijective-on-objects functor $b : A \rightarrow B$, and fully faithful functor $f : C \rightarrow D$, the square

$$\begin{array}{ccc} \mathbf{Cat}(B, C) & \xrightarrow{\mathbf{Cat}(b, C)} & \mathbf{Cat}(A, C) \\ \mathbf{Cat}(B, f) \downarrow & & \downarrow \mathbf{Cat}(A, f) \\ \mathbf{Cat}(B, D) & \xrightarrow{\mathbf{Cat}(b, D)} & \mathbf{Cat}(A, D) \end{array}$$

is a pullback in the category of sets. In fact the Gabriel factorisation is enriched over \mathbf{Cat} , meaning that these squares are also pullbacks in \mathbf{Cat} . Explicitly on arrows this says that given α and β as in

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ b \downarrow & \delta \searrow & \downarrow f \\ B & \xrightarrow{\beta} & D \end{array}$$

such that $f\alpha = \beta b$, there exists a unique δ as shown such that $\alpha = \delta b$ and $f\delta = \beta$. Moreover the Gabriel factorisation system is an *enhanced* factorisation system in the sense of Kelly [21], which is to say that given u, v and α as on the left,

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ b \downarrow & \alpha \cong & \downarrow f \\ B & \xrightarrow{v} & D \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{u} & C \\ b \downarrow & d \searrow & \downarrow f \\ B & \xrightarrow{v} & D \end{array}$$

there exists unique d and β as shown on the right, such that $u = db$ and $\alpha = \beta b$. It is not difficult to verify that the Gabriel factorisation system enjoys these properties [26].

From the explicit description of \mathbf{S} it is easy to see that \mathbf{S} preserves bijective-on-objects functors.

With these details in hand Power's idea boils down to the following. Given a symmetric strong monoidal functor $F : S \rightarrow M$ where S is a symmetric *strict* monoidal category, one can take the Gabriel factorisation

$$S \xrightarrow{G} M' \xrightarrow{H} M$$

of F , and then factor the coherence witnessing F as strong monoidal on the left

$$\begin{array}{ccc} SS & \xrightarrow{\otimes} & S \\ s(F) \downarrow & \overline{F} \cong & \downarrow F \\ SM & \xrightarrow{\otimes} & M \end{array} = \begin{array}{ccc} SS & \xrightarrow{\otimes} & S \\ s(G) \downarrow & = & \downarrow G \\ SM' & \xrightarrow{\otimes} & M' \\ s(H) \downarrow & \overline{H} \cong & \downarrow H \\ SM & \xrightarrow{\otimes} & M \end{array}$$

uniquely as on the right, using enhancedness and the fact that $\mathbf{S}G$ is bijective on objects.

Lemma A.4. [25] *In the situation just described, $\otimes : SM' \rightarrow M'$ is a symmetric strict monoidal structure on M' . With respect to this structure, G is a symmetric strict monoidal functor, and \overline{H} is the coherence datum making H into a symmetric strong monoidal functor.*

We attribute this result to Power, although it is not exactly formulated in this way in [25]. Power's result applies to more general monads T than \mathbf{S} (only required to preserve bijective-on-objects functors), but for the functor F he only considers the case where $S = TM$ and F is the action $TM \rightarrow M$. It is easy to adjust his argument to the present situation, to verify the strict \mathbf{S} -algebra axioms for $\otimes : SM' \rightarrow M'$, and the pseudo \mathbf{S} -morphism axioms for \overline{H} . Note that G is a strict \mathbf{S} -algebra morphism by construction.

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